BLOW UP IN SOME HYPERBOLIC PROBLEMS INVOLVING LEBESGUE AND SOBOLEV SPACES WITH VARIABLE EXPONENTS

BY

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This thesis, written by **ALA ATA TALAHMEH** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

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To my beloved parents, to my lovely wife and to my beautiful daughters, Mayaseen and Sadeen, I dedicate this work.

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THESIS ABSTRACT

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TITLE OF STUDY:	Blow Up in some Hyperbolic Problems Involving Lebesgue and Sobolev Spaces with Variable Exponents
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In this dissertation, we study the well-posedness as well as the blow up of some nonlinear hyperbolic problems involving nonclassical nonlinearities. In this regard, we prove several general blow up results under appropriate assumptions on the exponents of nonlinearity. We use the modified concavity method. We also exploit ideas by Messaoudi [72], Georgiev and Todorova [34] with appropriate modifications. Our results generalize some known results existing in the literature from the constant-exponent case to the variable-exponent case.

ملخص الرسالة

الإسم: علاء عطا عبد الهادي تلاحمة

عنوان الدراسة: خاصية الإنفجار لبعض المسائل الزائدية غيرالخطية التي تحتوي على حدود لاخطية غيركلاسيكية

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في هذه الأطروحة، نقوم بدراسة الصياغة الجيدة، فضلا عن خاصية الإنفجار، لبعض المسائل الزائدية غيرالخطية التي تنطوي على حدود لاخطية غيركلاسيكية. في هذا الصدد، نقوم بإثبات العديد من نتائج الإنفجار، تحت افتراضات مناسبة على أسس اللاخطية. نستخدم طريقة التقعر المعدّلة، كما نقوم بإستغلال بعض الأفكار من قبل مسعودي [72]، وجورجيف وتودوروفا [34] مع تعديلات مناسبة. نتائجنا تعمّم بعض النتائج المعروفة والموجودة في الدراسات من حالة الأس الثابت إلى حالة الأس المتغير.

CHAPTER 1

INTRODUCTION

1.1 History of Variable Exponent Spaces

Variable Lebesgue spaces were first introduced by W. Orlicz in 1931 in his article [81]. He started by looking for necessary and sufficient conditions on a sequence (y_i) in $\mathbb{R}$ under which $\sum x_i y_i$ converges, for any sequence (x_i) in $\mathbb{R}$ such that $\sum x_i^{p_i}$ converges, where (p_i) is a sequence of real numbers with $p_i > 1$. Furthermore, he also considered the variable exponent function space $L^{p(\cdot)}$ on the real line. Orlicz later concentrated much on the theory of the function spaces that were named after him (see [78]). In the theory of Orlicz spaces, the space L^{φ} is defined as follows:

$$L^{\varphi} := \Big\{ u : \Omega \to \mathbb{R} \text{ such that } \varrho(\lambda u) = \int_{\Omega} \varphi(\lambda |u(x)|) dx < +\infty \Big\},$$

for some $\lambda > 0$, where φ is a real-valued function that may depend on x and satisfies some additional conditions. Putting certain properties of ϱ in an abstract setting, a more general class of function spaces, called modular spaces, was first studied by Nakano [79],[80]. Following the work of Nakano, modular spaces were investigated by several people, most importantly by groups at Sapporo (Japan), Voronezh (U.S.S.R.), and Leiden (the Netherlands). An explicit version of modular function spaces was investigated by Polish Mathematicians, like Hudzik [40]-[50] and Kaminska [47],[56].

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the Orlicz space $L^{\varphi_{p(\cdot)}}(\Omega)$ where

$$\varphi_{p(\cdot)}(t) = t^{p(\cdot)} \quad \text{or} \quad \varphi_{p(\cdot)}(t) = \frac{t^{p(\cdot)}}{p(\cdot)},$$

i.e.,

$$L^{\varphi_{p(\cdot)}}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable such that } \varrho(\lambda u) = \int_\Omega \varphi_{p(x)}(\lambda|u(x)|)dx < +\infty\Big\},$$

for some $\lambda > 0$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 \text{ such that } \int_\Omega \varphi_{p(x)}\Big(\Big|\frac{u(x)}{\lambda}\Big|\Big)dx \leq 1\Big\}.$$

Variable-exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. Their results originated in 1961 in a paper by Tsenov [95]. The Luxemburg norm was introduced by Sharapudinov [90] for the Lebesgue space. He showed that this space is Banach if the exponent satisfies $1 < \text{essinf } p \leq \text{esssup } p < +\infty$. In the mid-80s, V. Zhikov [102] started a new line of investigation of variable-exponent spaces, by considering variational integrals with non-standard growth conditions. The next major step in the study of variable-exponent spaces was by Kovacik and Rakosnk [59] in the early 90's. In their paper, they established many of the basic properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$.

In the beginning of the new millennium, a great development has been made for the rigorous study of variable-exponent spaces. In particular, a connection was made between the variable-exponent spaces and the variational integrals with non-standard growth and coercivity conditions.

Recent systematic study of partial differential equations with variable exponents was motivated by the description of several relevant models in electrorheological fluids or fluids with temperature-dependent viscosity, thermorheological fluids, nonlinear viscoelasticity, filtration processes through a porous media and image processing, or robotics. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in gradient of the unknown solution and with variable exponents of nonlinearity. In what follows, we give an example due to Chen, Levine, Rao [20], which concerns application to image restoration. Consider an image u, recovered from an observed, noisy image I, where the two are related by

$$I = u + \text{ noise.}$$

Various types of diffusion, used for image restorations, arise from the minimization problem

$$\min_u \int_\Omega \Big[|\nabla u|^p + \frac{\lambda}{2}(u - I)^2 \Big], \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with Lipchitz boundary, $1 \leq p \leq 2$, $\lambda \geq 0$. There are two well known models. Namely, the isotropic diffusion ($p = 2$) and the total variation (TV)-based diffusion ($p = 1$).

TV minimization: The TV-based regularization, $p = 1$, does an excellent job of preserving edges while reconstructing images. Mathematically this is reasonable, since it is natural to consider solutions of this problem in the $BV(\Omega)$ space which allows discontinuities which are necessary for edge reconstruction. The TV model has proved to be a very efficient tool for preserving sharp edges.

Gaussian smoothing: Taking $p = 2$ results in an isotropic diffusion, which solves some problems but alone is not good for image reconstruction since it does preserve edges.

To combine the advantages and incorporates the strengths of the above models, many modifications have been introduced. For instance, Chambolle and Lions [19] proposed minimizing the following energy functional, which combines isotropic and TV-based diffusion

$$\min_{u \in BV(\Omega)} \frac{1}{2\beta} \int_{|\nabla u| \leq \beta} |\nabla u|^2 + \int_{|\nabla u| > \beta} |\nabla u| - \frac{\beta}{2}. \tag{1.2}$$

Functional with variable exponent $1 \leq p(x) \leq 2$: Chen et al. [20], in 2006, proposed a model which capitalizes on the strengths of (1.1) for different values of $1 \leq p(x) \leq 2$. This ensures a TV-based diffusion ($p \equiv 1$) along edges and a Gaussian smoothing ($p \equiv 2$) in homogeneous regions. In addition, it employs anisotropic diffusion $1 < p(x) < 2$ in piecewise smooth regions or where the difference between the noise and edges is difficult to distinguish. By taking $p = p(x)$, the direction and speed of diffusion at each location will depend on the local behavior. Precisely, they considered

$$\min_{u \in BV \cap L^2(\Omega)} \int_\Omega \phi(x, \nabla u) + \frac{\lambda}{2}(u - I)^2, \tag{1.3}$$

where

$$\phi(x, r) := \begin{cases} \frac{1}{q(x)} |r|^{q(x)}, & |r| \le \beta, \\ |r| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)}, & |r| > \beta, \end{cases}$$

for $\beta > 0,\ 1 < \alpha \le q(x) \le 2$. An example is

$$q(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I(x)|^2},$$

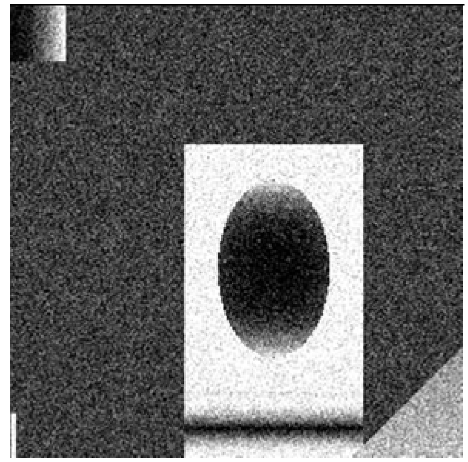
for fixed parameters $\sigma, k > 0$ and $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/4\sigma^2)$ is the Gaussian filter. The main features of this model is the way in which it accommodates the local image information. In fact, when the gradient is sufficiently large (at likely edges), only the TV-based diffusion will be used. But, when the gradient is near zero (in homogeneous regions), the model is isotropic. At all other locations, the filtering is somewhere between the Gaussian and the TV-based diffusion. In particular, the type of anisotropy at these ambiguous regions varies according to the strength of the gradient.

The associated flow with (1.3) is the following diffusion problem

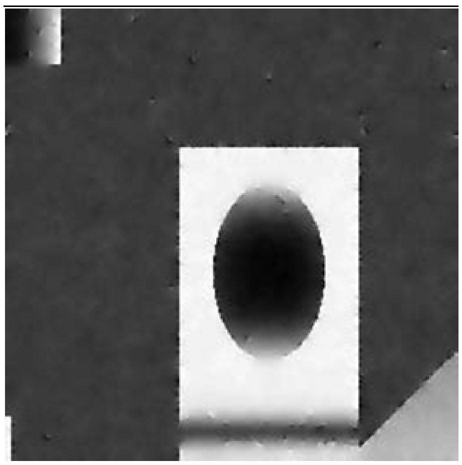
$$\begin{cases} u_t - \operatorname{div}(\nabla\varphi(x, \nabla u)) + \lambda(u - I) = 0, & \text{in } \Omega \times (0, T) \\ u(x, t) = g(x), & \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = I(x), & \text{in } \Omega. \end{cases} \tag{1.4}$$

For problem (1.3), Chen et al. [20] established the existence and uniqueness of the solution and the long-time behavior of the associated flow (1.4) of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in following figures.

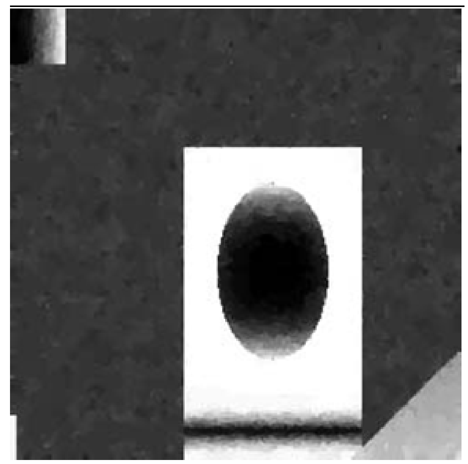
(a) Original image



(b) Image with noise

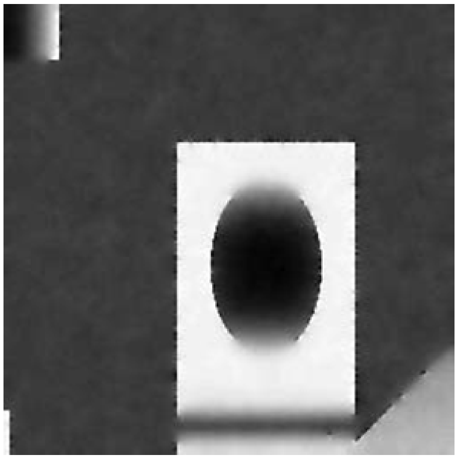


(c) reconstructions using (1.2) with thresholds $\beta = 30$ (1000 iterations).

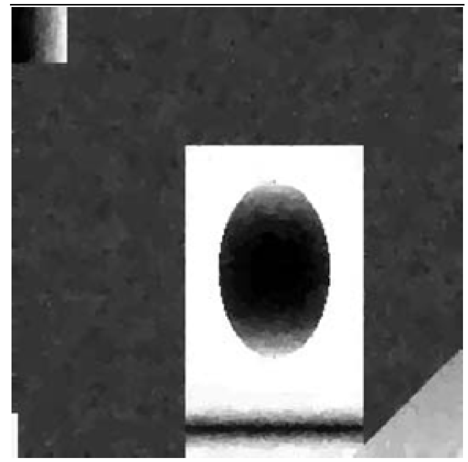


(d) reconstruction using TV-based diffusion only (2000 iterations)

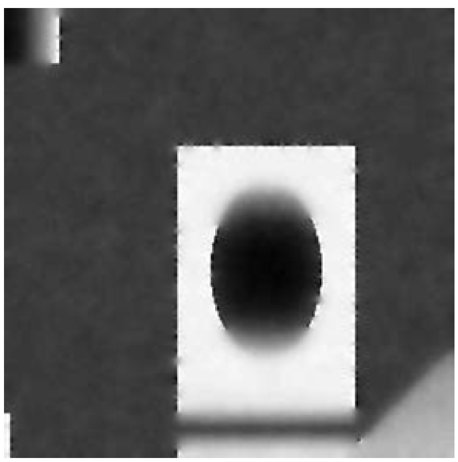
Figure 1.1: Experimental results I



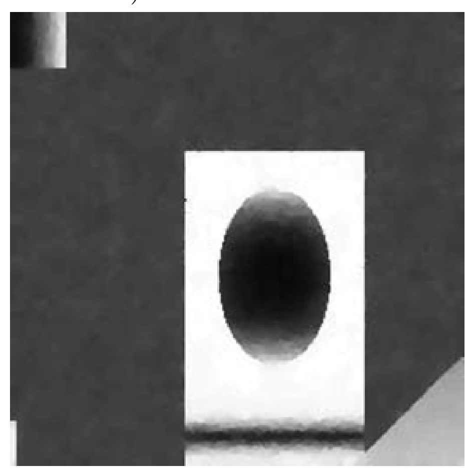
(a) reconstructions using (1.2) with thresholds $\beta = 50$



(b) reconstruction using the proposed model with thresholds $\beta = 30$ (1000 iterations).



(c) reconstructions using (1.2) with thresholds $\beta = 70$.



(d) reconstruction using the proposed model with thresholds $\beta = 100$ (1000 iterations).

Figure 1.2: Experimental results II

For more applications, the reader is referred to [1]-[3], [22], [27], [37], [89].

1.2 Results Description

The aim of this dissertation is to investigate the well-posedness as well as the blow up of solutions of some nonlinear hyperbolic problems involving nonclassical nonlinearities. In this regard, we study some problems and establish blow up results under some suitable assumptions. This study extends and generalizes several results. In particular, we extended the blow up result of some nonlinear hyperbolic problem, considered by Georgiev and Todorova [34] and Messaoudi [72], from the constant-exponent case to the variable-exponent case. Also, we extended, using the modified concavity method presented in [58], to problems with variable-exponent nonlinearities.

Our contributions start from Chapter three, where we investigate the existence and blow up of solutions of the following nonlinear wave problem with variable exponents

$$\begin{cases} u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, and $a, b \geq 0$ are constants and the exponents $m(\cdot)$ and $p(\cdot)$ are given log-Hölder continuous functions on Ω such that:

$$2 \leq m_1 \leq m(x) \leq m_2 \leq \frac{2n}{n-2}, \quad n \geq 3 \tag{1.6}$$

and

$$2 \leq p_1 \leq p(x) \leq p_2 \leq 2\frac{n-1}{n-2}, \quad n \geq 3. \tag{1.7}$$

We establish the well-posedness using the standard Faedo-Galerkin method and obtain the blow-up result for the solution under some suitable assumptions and give a two-dimension numerical example to illustrate the blow up result.

In Chapter four, we study the finite time blow-up of solutions of the following quasilinear wave equation with variable exponents:

$$\begin{cases} u_{tt} - \operatorname{div}\big(|\nabla u|^{r(\cdot)-2}\nabla u\big) + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $a, b > 0$ are constants, Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, and the exponents $m(\cdot), p(\cdot)$ and $r(\cdot)$ are given log-Hölder continuous functions on Ω such that:

$$2 \leq \max\{m_2, r_2\} < p_1 \leq p(x) \leq p_2 \leq r_*(x), \tag{1.9}$$

where

$$r_*(x) = \begin{cases} \frac{nr(x)}{\text{esssup}_{x\in\Omega}(n-r(x))} & \text{if } r_2 < n \\ \\ +\infty & \text{if } r_2 \geq n \end{cases}$$

and

$$\text{essinf}_{x\in\Omega}(r^*(x) - p(x)) > 0.$$

We establish a finite-time blowup result for the solutions with negative initial energy and for certain solutions with positive energy.

In Chapter five, we study the blow up of solutions of the following nonlinear wave equation with variable exponents

$$\begin{cases} u_{tt} - \text{div}(|\nabla u|^{m(x)-2}\nabla u) + \mu u_t = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T) \\ \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,T) \\ \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.10}$$

where $\mu \geq 0$ is a constant, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, and the exponents $m(\cdot)$ and $p(\cdot)$ are given log-Hölder continuous function on $\overline{\Omega}$ such that:

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < m_*(x), \tag{1.11}$$

where

$$m_*(x) = \begin{cases} \frac{nm(x)}{\text{esssup}_{x\in\Omega}(n-m(x))} & \text{if } m_2 < n \\ \\ +\infty & \text{if } m_2 \geq n \end{cases}$$

and

$$\text{essinf}_{x\in\Omega}(m^*(x) - p(x)) > 0.$$

We prove, for arbitrary positive initial energy, a finite-time blow-up result using modified concavity method. This extends the result of [58], to problems with variable-exponent nonlinearities.

1.3 Methodology

We use, in chapter five, the modified concavity method to establish the desired blow up result of the solution. This method relies mostly on the construction of an appropriate function $\Phi \in C^2([0,T))$. To prove the blow up we show that $\Phi(t)$, satisfies

$$\Phi\Phi_{tt} - \alpha\Phi_t^2 + \gamma\Phi\Phi_t + \beta\Phi \geq 0, \quad \alpha > 1, \beta \geq 0, \gamma \geq 0, \tag{1.12}$$

$$\Phi(t) \geq 0, \quad \Phi(0) > 0,$$

and

$$\Phi_t(0) > \frac{\gamma}{\alpha-1}\Phi(0), \ \ \Big(\Phi_t(0) - \frac{\gamma}{\alpha-1}\Phi(0)\Big)^2 > \frac{2\beta}{2\alpha-1}\Phi(0). \tag{1.13}$$

Then, (1.12) and (1.13) give the desired blow up result

$$\limsup_{t \to T} \Phi(t) = +\infty.$$

In chapter three and four, we exploit ideas by Messaoudi [72], with necessary modifications due to the nature of our equations, to prove the blow up of solution. For the well-posedness, in chapter three, we employ the standard Galerkin method combined with Banach fixed-point argument [62].

1.4 Literature Review

1.4.1 Blow Up in the Case of Constant Exponents.

The first study of finite-time blow up of solutions of hyperbolic PDEs of the form

$$u_{tt} - \Delta u = f(u)$$

goes back to Ball [14] in 1977. The interaction between the damping and the source terms was later considered by Levine [63]-[65] for an equation of the form

$$u_{tt} - \Delta u + a u_t = f(u).$$

He introduced the concavity method and showed that solutions with negative initial energy blow up at finite time. This method was later improved by Kalantarov and Ladyzhenskaya [55] to accommodate more general cases. After twenty years, Georgiev and Todorova [34] extended Levine's result to the nonlinearly damped equation

$$u_{tt} - \Delta u + a|u_t|^m u_t = b|u|^p u, \text{ in } \Omega \times (0, \infty),$$

for $a, b, m, p > 0$. In their work, Georgiev and Todorova introduced a different method and determined appropriate relations between the nonlinearies in the damping and the source, for which there is either global existence or alternatively finite-time blow up. Precisely, they showed that solutions with negative energy exist globally in time if $m \geq p$ and blow up in finite time if $p > m$ and initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine et al. [66], Levine and Serrin [67], Levine and Park [68], and Messaoudi [72], [73]. In all these papers, the authors showed that no solution with negative or sufficiently negative energy can be extended on $[0, \infty)$, if the nonlinearity in the source dominates the damping effect ($p > m$). Messaoudi [74] extended the result of [72] to the viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau) d\tau + a u_t |u_t|^m = b|u|^\gamma u, \text{ in } \Omega \times (0, \infty)$$

where Ω is a bounded domain of $\mathbb{R}^n$. This result was later pushed, by the same author [74] to certain solutions with positive initial energy. A similar result was also obtained by Wu [99] using a different method. Kafini and Messaoudi [53] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = b|u|^{p-2}u, \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

and established a blow up result. This latter result was later pushed by the same authors in [52] to a system of the form

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = f_1(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta v(\tau)d\tau = f_2(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

In the absence of the viscoelastic terms, Agre and Rammaha [4] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{m-1}v_t = f_2(u,v), & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & \text{in } \Omega, \end{cases} \tag{1.14}$$

in a bounded domain $\Omega \subset \mathbb{R}^n,\ n = 1,2,3,....$ and $f_1(u,v),\ f_2(u,v)$ are nonlinear functions satisfying

$$f_1(u,v) = \frac{\partial F}{\partial u}(u,v) \text{ and } f_2(u,v) = \frac{\partial F}{\partial v}(u,v)$$

for

$$F(u,v) = a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}, \;\; p \geq 3, \; a > 1, \; b > 0.$$

They proved several results concerning local and global existence of a weak solution and showed that any weak solution with negative initial energy blows up in finite time. In the presence of the viscoelastic terms, Messaoudi and Said-Houari [77] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \int_0^t h(t-\tau)\Delta v(\tau)d\tau + |v_t|^{r-1}v_t = f_2(u,v), & \text{in } \Omega \times (0,\infty), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & \text{in } \Omega, \\ v(x,0) = v_0(x), \; v_t(x,0) = v_1(x), & \text{in } \Omega, \end{cases} \tag{1.15}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. They proved a global existence result of certain solutions with positive initial energy. For more results concerning nonexistence, we mention here the work of Vitillaro [96],[97] Todorova [93], Todorova and Vitillaro [94], Wang [98], Liu [70], Wu and Lin [100], and the very recent book of Alshin et al. [7]. For the nonlinear Kirchhoff-type problem of the form

$$\begin{cases} u_{tt} - \Big(\int_\Omega |D^m u|^2 dx \Big)^q \Delta u + u_t |u_t|^r = |u|^p u, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.16}$$

where $p, q, r \geq 0$, Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$ and a unit outer normal ν, several results concerning global existence and blow-up have been established; see in this regard [15]-[16],[83]-[84], and the references therein. Messaoudi and Said-Houari [75] considered the nonlinear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}\big(|\nabla u|^{\alpha-2} \nabla u\big) - \operatorname{div}\big(|\nabla u_t|^{\beta-2} \nabla u_t\big) + a|u_t|^{m-2} u_t = b|u|^{p-2} u,$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with a regular boundary. They proved, under appropriate conditions on $\alpha, \beta, p, m > 2$, a global nonexistence result for solutions associated with negative initial energy. Chen et al. [21] considered the following nonlinear $p-$Laplacian-wave equation

$$u_{tt} - \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) - \Delta u_t + q(x,u) = f(x)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $2 \leq p < n$ and f, q are given functions. They established global existence and uniqueness under appropriate conditions on the initial data and the functions f, q. They also discussed the long time behaviour of the solution. Ibrahim and Lyaghfouri [51] considered the following equation

$$u_{tt} - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = |u|^{p_*-2}u$$

in $\mathbb{R}^n$, $n \geq 3$, and $2 < p < n$, $p_* = \frac{pn}{n-p}$ is the critical Sobolev exponent. Under appropriate assumptions on the initial data, they proved the finite-time blowup of solutions and, hence, extended a result by Galaktionov and Pohozaev [29]. Ye [101] investigated the blowup property of solutions of a quasilinear hyperbolic system of equations and proved that certain solutions with positive initial energy explode in finite time and he also gave estimation for the solution lifespan. Recently, Kafini and Messaoudi [54] studied a nonlinear wave equation with damping and delay terms and showed, under suitable hypotheses on the initial data, that the solution energy explodes in a finite time. For more results, we refer the reader to [17], [30], [38], [88].

1.4.2 Blow Up in the Case of Variable-Exponent Nonlinearities.

In recent years, much attention has been paid to the study of mathematical models of hyperbolic, parabolic or elliptic equations which are nonlinear with variable exponents of nonlinearity. In fact, there are only few works in this direction, let us mention some of them. For instance, Antontsev [12] considered the equation

$$u_{tt} - \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u) - \alpha\Delta u_t = b(x,t)u|u|^{\sigma(x,t)-2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $\alpha > 0$ is a constant and a, b, p, σ are given functions. For specific conditions on a, b, p, σ, he proved some blowup results, for certain solutions with non positive initial energy. He also discussed the case when $\alpha = 0$ and established a blow up result. Subsequently, Antontsev [11] discussed the same equation and proved a local and a global existence of some weak solutions under certain hypotheses on the functions a, b, p, σ. He also established some blowup results for certain solutions having non positive initial energy. Guo and Gao [35] looked into the same problem of [11] and established several blowup results for certain solutions associated with negative initial energy. Precisely, they took $\sigma(x,t) = \sigma > 2$, a constant, and established a result of blowup in finite-time. For the case $\sigma(x,t) = \sigma(x)$, they claimed the same blow up result but no proof has been given. This work is considered to be an improvement for that of [11]. In [92], Sun et al. looked into the following equation

$$u_{tt} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u_t|u_t|^{q(x,t)-1} = b(x,t)u|u|^{p(x,t)-1}$$

in a bounded domain, with Dirichlet-boundary conditions, and established a blowup result for solutions with positive initial energy. They also gave lower and upper bounds for the blowup time and provided a numerical illustrations for their result. Alaoui et al. [6] considered the following nonlinear heat equation

$$\begin{cases} u_t(x,t) - \operatorname{div}\big(|\nabla u|^{m(x)-2}\nabla u\big) = u|u|^{p(x)-2} + f, \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \end{cases} \tag{1.17}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, and $m(\cdot)$ and $p(\cdot)$ are two continuous functions on $\overline{\Omega}$ satisfying

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < m_*(x),$$

with

$$m_*(x) = \begin{cases} \frac{nm(x)}{\operatorname{esssup}_{x\in\Omega}(n-m(x))} & \text{if } m_2 < n \\ +\infty & \text{if } m_2 \geq n \end{cases}.$$

They also assumed that $m(\cdot)$ is log-Hölder continuous and

$$\operatorname{essinf}_{x\in\Omega}(m^*(x) - p(x)) > 0.$$

Under these conditions on $m(\cdot)$, $p(\cdot)$ and for $f = 0$ they showed that any solution with nontrivial initial datum blows up in finite time. They also gave a two-dimension numerical examples to illustrate their result. Gao and Gao [32] considered the following nonlinear viscoelastic hyperbolic problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.18}$$

where $m(x),\ p(x)$ are continuous functions in Ω such that

$$1 < \inf_{x\in\Omega} m(x) \le m(x) \le \sup_{x\in\Omega} m(x) < +\infty, \quad 1 < \inf_{x\in\Omega} p(x) \le p(x) \le \sup_{x\in\Omega} p(x) < +\infty$$

and

$$\forall z, \xi \in \Omega,\ |z-\xi| < 1,\ |m(z) - m(\xi)| + |p(z) - p(\xi)| \le \omega\big(|z-\xi|\big),$$

where

$$\limsup_{t\to 0^+} \omega(\tau)\ln(\frac{1}{\tau}) = C < +\infty.$$

They also assumed that

(i) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = \ell > 0;$$

(ii) There exists $\eta > 0$ such that

$$g'(t) \leq -\eta g(t), \quad t \geq 0.$$

They proved the existence and energy decay of the solutions to problem (1.18). Akagi and Matsuura [5] were interested in the initial-boundary value problem for a nonlinear parabolic equation involving the $p(x)-$Laplacian. Precisely, they considered the following problem:

$$\begin{cases} u_t(x,t) - \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x,t), \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \end{cases} \tag{1.19}$$

where $f : \Omega \times (0,\infty) \to \mathbb{R}$, $u_0 : \Omega \times (0,\infty) \to \mathbb{R}$ and $p(\cdot) : \Omega \to [1,\infty]$ are given functions. The well-posedness and the long-time behaviour of the L^2-solution to problem (1.19) were established, using the subdifferential calculus approach. A parabolic equation of the form (1.19) was also studied by Bendahmane et al. [18], where the well-posedness of a solution was proved for L^1-data. Ferreira and Messaoudi [28] studied a nonlinear viscoelastic plate equation with a lower order perturbation of a $\overrightarrow{p}(x,t)-$ operator of the form

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta_{\overrightarrow{p}(x,t)} u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + f(u) = 0, \text{in}\Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times [0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \end{cases} \tag{1.20}$$

where

$$\Delta_{\overrightarrow{p}(x,t)} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\Big), \quad \overrightarrow{p} = (p_1, p_2, ..., p_n)^T$$

and $g \geq 0$ is a memory kernel that decays at a general rate and $f(u)$ is a nonlinear function. They proved a general decay result for appropriate conditions on g, f and the variable exponent $\overrightarrow{p}(x,t)-$Laplacian operator. Autuori et al. [13] looked into a nonlinear Kirchhoff system in the presence of the $\overrightarrow{p}(x,t)-$Laplacian operator, a nonlinear force $f = f(t,x,u)$ and a nonlinear damping term $Q = Q(t,x,u,u_t)$. They established a global nonexistence result under suitable conditions on $f, Q, p(x)$. We refer the reader to Antontsev [12] and Galaktionov [31] for more problems involving the variable-exponent nonlinearities.

CHAPTER 2

PRELIMINARIES

2.1 Lebesgue Spaces with Variable Exponents

In this section, we present some preliminary facts about Lebesgue spaces with variable-exponents (see [24],[25],[27],[57],[60]).

Definition 2.1 The function ϱ is said to be *left-continuous* if the mapping $\lambda \longmapsto \varrho(\lambda x)$ is left-continuous on $[0, \infty)$ for every $x \in X$, that is,

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x).$$

Definition 2.2 Let X be a $\mathbb{K}-$vector space, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. A function $\varrho : X \longrightarrow [0, \infty]$ is called a *semimodular* on X if the following properties hold:

(a) $\varrho(0) = 0$;

(b) $\varrho(\lambda x) = \varrho(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| = 1$;

(c) ϱ is convex;

(d) ϱ is left-continuous;

(e) $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

A semimodular is called *modular* if

(f) $\varrho(x) = 0$ implies $x = 0$.

A semimodular is called *continuous* if

(g) the mapping $\lambda \longmapsto \varrho(\lambda x)$ is continuous on $[0, \infty)$ for all $x \in X$.

Examples 2.1.1 Let $L^0(\Omega)$ be the set of all Lebesgue measurable functions defined on Ω. If $1 \leq p < +\infty$, then

$$\varrho_p(f) := \int_\Omega |f(x)|^p dx$$

defines a continuous modular on $L^0(\Omega)$.

Theorem 2.1 [60] *Let ϱ be a semimodular on X. Then by convexity and non-negativeness of ϱ and $\varrho(0) = 0$, the mapping $\lambda \to \varrho(\lambda x)$ is non-decreasing on $[0, \infty)$ for every $x \in X$. Moreover*

$$\begin{aligned} \varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x) \quad \text{for all} \quad |\lambda| \leq 1, \\ \varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x) \quad \text{for all} \quad |\lambda| \geq 1. \end{aligned} \tag{2.1}$$

Proof.

- Assume that $0 \leq \lambda < \mu$, then $0 \leq \dfrac{\lambda}{\mu} < 1$. So for fixed $x \in X$ we have

$$\varrho(\lambda x) = \varrho\Big(\frac{\lambda}{\mu}(\mu x) + \big(1 - \frac{\lambda}{\mu}\big) \cdot 0\Big) \leq \frac{\lambda}{\mu}\varrho(\mu x) + \Big(1 - \frac{\lambda}{\mu}\Big)\varrho(0) = \frac{\lambda}{\mu}\varrho(\mu x) \leq \varrho(\mu x).$$

  Hence for any fixed $x \in X$, we have

$$\varrho(\lambda x) \leq \varrho(\mu x) \qquad \text{for} \qquad 0 \leq \lambda < \mu.$$

- For $\lambda \neq 0$ we have

$$\varrho(\lambda x) = \varrho\Big(\frac{\lambda}{|\lambda|}|\lambda| x\Big) = \varrho(|\lambda| x) \quad \Big(\text{since } \Big|\frac{\lambda}{|\lambda|}\Big| = 1\Big).$$

- For $|\lambda| \leq 1$, we have

$$\varrho(|\lambda| x) = \varrho(|\lambda| x + (1 - |\lambda|)0) \leq |\lambda|\varrho(x) + (1 - |\lambda|)\varrho(0) = |\lambda|\varrho(x).$$

  Therefore,

$$\varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda|\varrho(x) \qquad \forall x \in X \qquad \text{and} \qquad |\lambda| \leq 1.$$

- For $|\lambda| \geq 1$, we have

$$\varrho(x) = \varrho\Big(\frac{1}{|\lambda|}|\lambda| x\Big) \leq \frac{1}{|\lambda|}\varrho(|\lambda| x) + \Big(1 - \frac{1}{|\lambda|}\Big)\varrho(0) = \frac{1}{|\lambda|}\varrho(|\lambda| x).$$

Thus,

$$\varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x) \qquad \forall x \in X \qquad \text{and } |\lambda| \geq 1.$$

■

Definition 2.3 Let (Ω, Σ, μ) be a $\sigma-$finite, complete measure space. We define $\mathcal{P}(\Omega, \mu)$ to be the set of all $\mu-$measurable functions $p : \Omega \to [1, \infty]$. The functions $p \in \mathcal{P}(\Omega, \mu)$ are called variable exponents on Ω. We define

$$p_1 := \text{essinf}_{y \in \Omega} p(y) \;\text{ and }\; p_2 := \text{esssup}_{y \in \Omega} p(y).$$

If $p_2 < \infty$, then we call p a bounded variable exponent. If $p \in \mathcal{P}(\Omega, \mu)$, then we define $p' \in \mathcal{P}(\Omega, \mu)$ by

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \;\text{ where }\; \frac{1}{\infty} := 0.$$

The function p' is called the dual variable exponent of p. In the special case that μ is the $n-$ dimentional Lebesgue measure and Ω is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$.

Definition 2.4 We define the Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{p(\cdot)}(\lambda u) < \infty,\ \text{for some } \lambda > 0 \Big\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx$$

is a modular, equipped with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} := \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\}.$$

Examples 2.1.2 Let $p(x) = x$ on $(1, 2)$. Then $\|1\|_{p(\cdot)} = 1$. Indeed,

$$\varrho_{p(\cdot)}(1/\lambda) = \int_1^2 \lambda^{-x} dx = \frac{\lambda - 1}{\lambda^2 \ln \lambda}.$$

Since $\varrho_{p(\cdot)}(1) = 1$, then, by definition of $\|1\|_{p(\cdot)}$, we have $\|1\|_{p(\cdot)} \leq 1$. On the other hand, it is easy to check that $\varrho_{p(\cdot)}(1/\lambda) > 1$, for $0 < \lambda < 1$. This gives $\|1\|_{p(\cdot)} > 1$. Hence, we conclude that $\|1\|_{p(\cdot)} = 1$.

Lemma 2.1 If $p(x) \equiv p$, where p is constant. Then,

$$\|u\|_{p(\cdot)} = \lambda_0 = \Big(\int_\Omega |u|^p dx \Big)^{\frac{1}{p}}. \tag{2.2}$$

Proof. Since $\varrho_{p(\cdot)}(u/\lambda_0) = 1$, then

$$\|u\|_{p(\cdot)} \leq \lambda_0. \tag{2.3}$$

Next, using property of Inf, there exists a sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that $\lambda_k \geq \|u\|_{p(\cdot)}$, with

$$\varrho_{p(\cdot)}(u/\lambda_k) \leq 1 \ \text{ and } \ \lambda_k \to \|u\|_{p(\cdot)}.$$

Since, $\varrho_{p(\cdot)}(u/\lambda_k) = \frac{1}{(\lambda_k)^p}\int_\Omega |u|^p \leq 1$, then we have

$$\lambda_0 \leq \|u\|_{p(\cdot)}. \tag{2.4}$$

Combining (2.3) and (2.4) gives (2.2). ∎

Definition 2.5 We say that a function $q : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω if

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \ \text{ with } |x-y| < \delta, \tag{2.5}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.2 *Let Ω be a domain of $\mathbb{R}^n$. If $p : \Omega \to \mathbb{R}$ is a Lipchitz function, then it is log-Hölder continuous on Ω.*

Proof. Let $x, y \in \Omega$, with $|x-y| < \delta$ and $0 < \delta < 1$. Then, since p is Lipchitz, there exists $L > 0$ such that

$$\begin{aligned} |p(x) - p(y)| &\leq L|x-y| \\ &\leq -\frac{L}{\log|x-y|}\big(-|x-y|\log|x-y|\big). \end{aligned} \tag{2.6}$$

Let $g(s) = -s\log s$. Then, g is continuous on $[0,1]$ and hence is bounded. So we have, $0 \leq -s \log s \leq M$. Therefore, (2.6) becomes

$$|p(x) - p(y)| \leq -\frac{A}{\log |x-y|},$$

where $A = LM > 0$. Hence, p is log-Hölder continuous $\blacksquare$

Examples 2.1.3 Let $p(x,y) = x^2 + 1$ be defined on $\Omega = B(0,1)$. Then, by previous lemma, $p : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω.

Lemma 2.3 (Unit ball property) [60] *Let $p \in \mathcal{P}(\Omega, \mu)$ and $f \in L^{p(\cdot)}(\Omega, \mu)$. Then*

(i) $\|f\|_{p(\cdot)} \leq 1$ if and only if $\varrho_{p(\cdot)}(f) \leq 1$.

(ii) If $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$.

(iii) If $\|f\|_{p(\cdot)} \geq 1$, then $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$.

(iv) $\|f\|_{p(\cdot)} \leq 1 + \varrho_{p(\cdot)}(f)$.

Proof.

(i) If $\varrho_{p(\cdot)}(f) \leq 1$, then $\|f\|_{p(\cdot)} \leq 1$ by definition of $\|\cdot\|_{p(\cdot)}$. On the other hand, if $\|f\|_{p(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1$ for all $\lambda > 1$. Since $\varrho_{p(\cdot)}$ is left-continuous it follows that $\varrho_{p(\cdot)}(f) \leq 1$.

(ii) The claim is obvious for $f = 0$, so assume that $0 < \|f\|_{p(\cdot)} \leq 1$. By (i) and $\big\|f/\|f\|_{p(\cdot)}\big\| = 1$, it follows that $\varrho_{p(\cdot)}\big(f/\|f\|_{p(\cdot)}\big) \leq 1$. Since $\|f\|_{p(\cdot)} \leq 1$, it follows from (2.1) that $\varrho_{p(\cdot)}(f)/\|f\|_{p(\cdot)} \leq 1$. This implies $\varrho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$.

(iii) Assume that $\|f\|_{p(\cdot)} > 1$. Then $\varrho_{p(\cdot)}\big(\frac{f}{\lambda}\big) > 1$ for $1 < \lambda < \|f\|_{p(\cdot)}$ and by (2.1) it follows that $1 < \frac{\varrho_{p(\cdot)}(f)}{\lambda}$. which implies $\lambda < \varrho_{p(\cdot)}(f)$, for $1 < \lambda < \|f\|_{p(\cdot)}$. Since λ is arbitrary, we have $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$.

(iv) This follows immediately from (ii) and (iii).

$\blacksquare$

Lemma 2.4 [60] *If $1 < p_1 \leq p(x) \leq p_2 < \infty$ holds, then*

$$\min\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\} \leq \varrho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$.

Theorem 2.2 [60] *If $p \in \mathcal{P}(\Omega, \mu)$, then $L^{p(\cdot)}(\Omega, \mu)$ is a Banach space.*

Lemma 2.5 [60] *If $p : \Omega \to [1, \infty)$ is a measurable function with $p_2 < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Lemma 2.6 (Young's Inequality) [60] *Let $p, q, s \in \mathcal{P}(\Omega, \mu)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for } \mu- \textit{ a.e } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \leq \frac{(a)^{p(\cdot)}}{p(\cdot)} + \frac{(b)^{q(\cdot)}}{q(\cdot)}. \tag{2.7}$$

By taking p, q, s constants such that $s = 1$, and $1 < p, q < \infty$, then we have for any $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \forall a, b \geq 0,$$

where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. For $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Lemma 2.7 (Hölder's Inequality) [60] *Let $p, q, s \in \mathcal{P}(\Omega, \mu)$ such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \textit{for } \mu\textit{−a.e } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega, \mu)$ and $g \in L^{q(\cdot)}(\Omega, \mu)$, then $fg \in L^{s(\cdot)}(\Omega, \mu)$ and

$$\|fg\|_{s(\cdot)} \leq \; 2 \; \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

By taking $p = q = 2$, we have the ***Cauchy-Schwarz inequality****.*

2.2 Sobolev Spaces with Variable Exponents

In this section we study functional analysis-type properties of Sobolev spaces with variable exponents. We start by recalling the definition of weak derivative.

Definition 2.6 (weak derivative). Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $|\alpha| = \alpha_1 + ... + \alpha_n$.

If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{|\alpha|}\psi}{\partial^{\alpha_1}x_1.....\partial^{\alpha_n}x_n} dx = (-1)^{|\alpha|} \int_\Omega \psi g \, dx$$

for all $\psi \in C_0^\infty(\Omega)$, then g is called a *weak partial derivative* of u of order α. The function g is denoted by $\partial_\alpha u$ or $\frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1.....\partial^{\alpha_n}x_n}$.

Definition 2.7 Let $k \in \mathbb{N}$. We define the space $W^{k,p(\cdot)}(\Omega)$ by

$$W^{k,p(\cdot)}(\Omega) := \Big\{ u \in L^{p(\cdot)}(\Omega) \text{ such that } \partial^{|\alpha|}u \in L^{p(\cdot)}(\Omega) \text{ with } |\alpha| \le k \Big\}.$$

We define a semimodular on $W^{k,p(\cdot)}(\Omega)$ by

$$\varrho_{W^{k,p(\cdot)}(\Omega)}(u) = \sum_{0 \le |\alpha| \le k} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u).$$

This induces a norm *[60]* given by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf \Big\{ \lambda > 0 : \varrho_{W^{k,p(\cdot)}(\Omega)}\Big(\frac{u}{\lambda}\Big) \le 1 \Big\} = \sum_{0 \le |\alpha| \le k} \|\partial_\alpha u\|_{p(\cdot)}.$$

For $k \in \mathbb{N}$, the space $W^{k,p(\cdot)}(\Omega)$ is called Sobolev space and its elements are called *Sobolev functions.* Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$, and

$$W^{1,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) \ \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \Big\},$$

with a norm given by $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$. We abbreviate $\|u\|_{W^{k,p(\cdot)}(\Omega)}$

to $\|u\|_{k,p(\cdot)}$ and $\varrho_{W^{k,p(\cdot)}(\Omega)}$ to $\varrho_{k,p(\cdot)}$.

Theorem 2.3 [60] *Let $p \in \mathcal{P}(\Omega)$. The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive if $1 < p_1 \leq p_2 < \infty$.*

Definition 2.8 Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W_0^{k,p(\cdot)}(\Omega)$ with zero trace is the closure of the set of $W^{k,p(\cdot)}(\Omega)-$functions with compact support, i.e.,

$$W_0^{k,p(\cdot)}(\Omega) = \overline{\left\{u \in W^{k,p(\cdot)}(\Omega) : u = u\chi_K \ for \ a \ compact \ K \subset \Omega\right\}}$$

in $W^{k,p(\cdot)}(\Omega)$.

Remark 2.1 [60] *Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. Then*

(i) The space $H_0^{k,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.

(ii) $H_0^{k,p(\cdot)}(\Omega) \subset W_0^{k,p(\cdot)}(\Omega)$.

(iii) If p is log-Hölder continuous on Ω, then $W_0^{k,p(\cdot)}(\Omega) = H_0^{k,p(\cdot)}(\Omega)$.

(iv) The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W^{-1,p'(\cdot)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Theorem 2.4 [60] *Let $p \in \mathcal{P}(\Omega)$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive if $1 < p_1 \leq p_2 < \infty$.*

Theorem 2.5 (**Poincaré's inequality**). [60] *Let Ω be a bounded domain of $\mathbb{R}^n$ and suppose that $p(\cdot)$ is log-Hölder continuous on Ω, then*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \quad for \ all \ \ u \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant C depends on p_1, p_2 and Ω only. In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$.
If $p = 2$; then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Remark 2.2 The log-Hölder continuity condition on $p(\cdot)$ can be replaced by $p(\cdot) \in C(\overline{\Omega})$, if Ω is bounded.

Remark 2.3 Contrary to the constant-exponent case, there is no Poincaré inequality version for modular. The following example shows that the Poincaré inequality does not, in general, hold in a modular form.

Examples 2.2.1 [60] Let $p : (-2, 2) \longrightarrow [2, 3]$ be a Lipschitz continuous exponent defined by

$$p(x) = \begin{cases} 3, & if\ x \in (-2,-1) \cup (1,2) \\ 2, & if\ x \in (-\frac{1}{2}, \frac{1}{2}) \\ -2x+1, & if\ x \in [-1, -\frac{1}{2}] \\ 2x+1, & if\ x \in [\frac{1}{2}, 1]. \end{cases}$$

Let u_μ be a Lipschitz function defined by

$$u_\mu(x) = \begin{cases} \mu x + 2\mu, & if\ x \in (-2,-1] \\ \mu, & if\ x \in (-1,1) \\ -\mu x + 2\mu, & if\ x \in [1,2). \end{cases}$$

Then

$$\frac{\varrho(u_\mu)}{\varrho(u'_\mu)} = \frac{\int_{-2}^{2} |u_\mu|^{p(x)}\, dx}{\int_{-2}^{2} |u'_\mu|^{p(x)}\, dx} \geq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \mu^2\, dx}{2\int_{-2}^{-1} \mu^3\, dx} = \frac{1}{2\mu} \to \infty$$

as $\mu \to 0^+$.

We end this section with some essential embedding results which are needed for the proofs in this dissertation.

Lemma 2.8 [60] *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume that $p : \Omega \to (1, \infty)$ is a measurable function such that*

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \quad \textit{for a.e. } x \in \Omega.$$

If $p(x), q(x) \in C(\overline{\Omega})$ and $q(x) < p^*(x)$ in $\overline{\Omega}$ with $p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & \textit{if } p_2 < n \\ \infty, & \textit{if } p_2 \geq n, \end{cases}$ *then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

As a special case we have

Corollary 2.6 [60] *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume that $p(\cdot) \in C(\overline{\Omega})$ such that*

$$2 \leq p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2}, \quad n \geq 3. \tag{2.8}$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

Remark 2.4 The $p(\cdot)-$Laplacian with a variable exponent $p(\cdot)$ is related to the variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega)$ whose theory has been rigorously studied in [60]. Equation (1.19) in Section 1.6 of the last chapter is a nonlinear diffusion equation which has been used to study image restoration and elecrtoheological fluids (see [2], [3], [22], [26], [27] and references therein).

2.3 Notations

Throughout this dissertation, we use the standard $L^2(\Omega)$ and $H^1(\Omega)$ spaces. The space $H^1(\Omega)$ is equipped with the norm

$$\|u\|^2_{H^1(\Omega)} = \|u\|_2 + \|\nabla u\|_2,$$

where $\|u\|_2 = \|u\|^2_{L^2(\Omega)}$. We also make use of the space

$$H^1_0(\Omega) = \{u \in H^1(\Omega) \colon \exists \{u_m\}_{m=0}^{\infty} \subset C^1_0(\Omega), \text{ such that } u_m \to u \text{ in } H^1(\Omega)\},$$

equipped with the norm

$$\|u\|^2_{H^1_0(\Omega)} = \|\nabla u\|^2_2,$$

if Ω is a bounded domain. The following standard notations are used in the dissertation

- $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + ... + \partial^2_{x_n}$
- $\nabla = (\partial_{x_1},\ \partial_{x_2}, ..., \partial_{x_n})$

- $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2},$

- $C^1(\Omega)$ denotes the space of all continuously differentiable functions on Ω,

- $C_0^1(\Omega)$ denotes the space of all continuously differentiable functions with compact support in Ω. The support of a continuous function f defined on Ω is the closure of the set of point where $f(x)$ is nonzero. That is

$$\text{supp}(f) :=\overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

- $C_0^\infty(\Omega)$ denotes the space of all continuously functions with compact support in Ω, having continuous derivatives of all orders.

- $L^p(\Omega) = \{f : \Omega \to \mathbb{R};\ f$ is measurable function and $\int_\Omega |f|^p dx < +\infty\}$, where $1 \leq p < \infty$.

- $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R};\ f$ is measurable function and there is a constant $C \geq 0$ such that $|f(x)| \leq C$ a.e. on $\Omega\}$.

- $L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R};\ f$ is measurable function and $f \in L^p(K),\ \forall\, K \subset \Omega, K$ compact$\}$.

Let X be a real Banach space with a norm $\|\cdot\|$. We have the following definitions

- The space $L^p(0,T;X)$ consists of all measurable functions $u : [0,T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \Big(\int_0^T \|u(t)\|^p dt\Big)^{1/p} < +\infty,$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(0,T;X)} := \text{esssup}_{0\leq t\leq T}\|u(t)\| < +\infty,$$

for $p = \infty$.

- The space $L^p_{loc}(0,T;X)$ consists of all measurable functions $u : (0,T) \to X$ with $u \in L^p([a,b];X)$ for every closed interval $[a,b] \subset (0,T)$.

- The space $C([0,T],X)$ consists of all continuous functions $u : [0,T] \to X$ with

$$\|u\|_{C([0,T],X)} := \max_{0\leq t\leq T} \|u\| < +\infty$$

.

- The space $C^1([0,T],X)$ consists of all continuously differentiable functions $u : [0,T] \to X$ with

$$\|u\|_{C^1([0,T],X)} := \max_{0\leq t\leq T} \|u\| + \max_{0\leq t\leq T} \|\frac{du}{dt}\| < +\infty.$$

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Publications

The following papers were Published/Submitted from our research

(1) Messaoudi S.A. and Talahmeh A.A, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities,* Applicable Analysis 96 no. 9 (2017), 1509-1515.

(2) S.A. Messaoudi, A.A. Talahmeh and J.H. Al-Smail, *Nonlinear Damped Wave Equation: Existence and Blow-up*, Journal of Computers and Mathematics with Applications 74 no. 12 (2017), 3024-3041.

(3) Messaoudi S.A and Talahmeh A.A, *A blow-up result for a quasilinear wave equation with variable-exponent nonlinearities,* Math Meth Appl Sci. (2017), 1-11. https://doi.org/10.1002/mma.4505 (Appeared).

(4) Messaoudi S.A. and Talahmeh A.A., *On wave equation: Review and recent results,* Arabian Journal of Mathematics (AJOM). DOI:10.1007/s40065-017-0190-4 (Appeared).

(5) Al-Smail J.H., Messaoudi S.A and Talahmeh A.A., *Well-posedness and numerical study for solutions of parabolic equation with variable-exponent nonlinearities,* International Journal of Differential Equations (Accepted).

(6) Messaoudi S.A., Al-Smail J.H. and Talahmeh A.A., *Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities,* (Submitted).

CHAPTER 3

NONLINEAR DAMPED WAVE EQUATION: EXISTENCE AND BLOW-UP

In this chapter, we prove the well-posedness and the finite time blow-up of solutions of the following nonlinear wave problem with variable exponents:

$$\begin{cases} u_{tt} - \Delta u + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $a, b > 0$ are constants and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, and the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions defined on Ω. In Section 3.1, we introduce some assumptions needed in this chapter. In Section 3.2, we use the Galerkin method to prove the well-posedness of the problem. Some technical lemmas are given in Section 3.3. The statement with the proof of our main blow up result will be given in Section 3.4. In Section 3.5, we present a two-dimension numerical example to illustrate the blow up result.

3.1 Assumptions

In this section, we present some materials needed in the proof of our results and establish the well-posedness of the problem. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and assume the following hypotheses

$(A1)$ The exponents m and p are measurable functions such that either $m, p \in C(\overline{\Omega})$ or they satisfy the following log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \quad \text{with } |x-y| < \delta. \tag{3.2}$$

$A > 0, \quad 0 < \delta < 1.$

$(A2)$ For the nonlinearity in the damping, we assume that

$$\begin{aligned} &2 \leq m_1 \leq m(x) \leq m_2 \leq \frac{2n}{n-2}, \quad n \geq 3. \\ &2 \leq m_1 \leq m(x) \leq m_2 < +\infty, \quad n < 3. \end{aligned} \tag{3.3}$$

$(A3)$ For the nonlinearity in the source term, we assume that

$$\begin{aligned} &2 \le p_1 \le p(x) \le p_2 \le 2\frac{n-1}{n-2}, \quad n \ge 3. \\ &2 \le p_1 \le p(x) \le p_2 < +\infty, \quad n < 3. \end{aligned} \tag{3.4}$$

$(A4)$ For the blow-up result, we further assume that

$$2 \le m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 \le \frac{2n}{n-2}. \tag{3.5}$$

We introduce the energy associated to problem (3.1)

$$E(t) := \frac{1}{2}\int_\Omega \left[u_t^2 + |\nabla u|^2\right] dx - b\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx, \quad t \ge 0. \tag{3.6}$$

Direct differentiation, using problem (3.1), leads to

$$E'(t) = -a\int_\Omega |u_t(x,t)|^{m(x)} dx, \quad \text{for a.e } t \in [0,T). \tag{3.7}$$

3.2 The Well-posedness of the Problem

In this section, we give the existence and uniqueness result for problem (3.1), using the Galerkin method combined with a fixed-point argument. First, we consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = f(x,t), & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{3.8}$$

where $a > 0$ is a constant, $f \in L^2\big(\Omega \times (0,T)\big)$, $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the exponent m is a given measurable function satisfying $(A1) - (A2)$, and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

Proposition 3.1 *Let* $m \in \mathcal{C}(\overline{\Omega})$. *Under condition* $(A2)$, *problem* (3.8) *has a unique local solution*

$$u \in L^\infty\big((0,T), H_0^1(\Omega)\big), \;\; u_t \in L^\infty\big((0,T), L^2(\Omega)\big) \cap L^{m(\cdot)}\big(\Omega \times (0,T)\big),$$

$$u_{tt} \in L^2\big((0,T), H^{-1}(\Omega)\big).$$

Proof. ***Uniqueness:***

Suppose that (3.8) *has two solutions* u *and* v. *Then,* $w = u - v$ *satisfies*

$$\begin{cases} w_{tt} - \Delta w + a u_t |u_t|^{m(\cdot)-2} - a v_t |v_t|^{m(\cdot)-2} = 0, & \textit{in } \Omega \times (0,T) \\ w(x,t) = 0, & \textit{on } \partial\Omega \times (0,T) \\ w(x,0) = w_t(x,0) = 0, & \textit{in } \Omega. \end{cases}$$

Multiply by w_t and integrate over Ω, to obtain

$$\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega w_t^2 + \int_\Omega |\nabla w|^2\Big] + a\int_\Omega \big(u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\big)\big(u_t - v_t\big)dx = 0.$$

Integrate over $(0,t)$, to get

$$\int_\Omega \big(w_t^2 + |\nabla w|^2\big) + 2a\int_0^t\int_\Omega \big(u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\big)\big(u_t - v_t\big)dxds = 0$$

By using the inequality

$$\big(|a|^{m(x)-2}a - |b|^{m(x)-2}b\big)(a-b) \geq 0,$$

for all $a, b \in \mathbb{R}$ and a.e. $x \in \Omega$, we have

$$\int_\Omega \big(w_t^2 + |\nabla w|^2\big) = 0;$$

which implies that $w = C = 0$, since $w = 0$ on $\partial\Omega$. Hence, the uniqueness.

Existence:

Let $\{v_j\}_{j=1}^\infty$ be an orthonormal basis of $H_0^1(\Omega)$, with

$$-\Delta v_j = \lambda_j v_j, \quad in \;\; \Omega, \quad v_j = 0, \quad on \;\; \partial\Omega,$$

and define the finite-dimensional subspace $V_k = span\{v_1, ..., v_k\}$. By normalization, *we have* $\|v_j\|_2 = 1$. *We look for functions*

$$u^k(x,t) = \sum_{j=1}^{k} a_j(t) v_j, \quad x \in \Omega, \ \ t \in (0,T)$$

which satisfy the following approximate problems

$$\begin{aligned} &\int_\Omega u^k_{tt}(x,t) v_j(x) dx + \int_\Omega \nabla u^k(x,t) \nabla v_j(x) dx \\ &+ a \int_\Omega |u^k_t(x,t)|^{m(x)-2} u^k_t(x,t) v_j(x) dx = \int_\Omega f(x,t) v_j(x) dx, \\ &u^k(x,0) = u^k_0 \ \ , \ \ u^k_t(x,0) = u^k_1, \ \ \forall j = 1,2,....,k, \ t \in (0,T) \end{aligned} \tag{3.9}$$

where $u^k_0 = \sum_{i=1}^{k} (u_0, v_i) v_i$, $u^k_1 = \sum_{i=1}^{k} (u_1, v_i) v_i$ *are two sequences in* $H^1_0(\Omega)$ *and* $L^2(\Omega)$, *respectively, such that*

$$u^k_0 \to u_0 \ \ in \ \ H^1_0(\Omega) \ \ and\, u^k_1 \to u_1 \ \ in \ \ L^2(\Omega).$$

This generates the system of k *ordinary differential equations*

$$\begin{cases} a''_j(t) + \lambda_j a_j(t) = g_j(t) + G_j(a'_1(t), ..., a'_k(t)), \\ \\ a_j(0) = (u_0, v_j), a'_j(0) = (u_1, v_j), \ \ \forall j = 1,2,....,k, \ t \in (0,T), \end{cases} \tag{3.10}$$

where

$$g_j(t) = \int_\Omega f(x,t) v_j(x) dx, \ t \in (0,T)$$

and

$$G_j(a_1'(t), ..., a_k'(t)) = -a \int_\Omega \big| \sum_{i=1}^{k} a_i'(t) v_i(x) \big|^{m(x)-2} \sum_{i=1}^{k} a_i'(t) v_i(x) v_j(x) dx.$$

This system can be solved by standard ODE theory. Hence, we obtain functions

$$a_j : [0, t_k) \to \mathbb{R},\ 0 < t_k < T.$$

Next, we have to show that $t_k = T,\ \forall k \geq 1$. *We multiply (3.9) by* $a_j'(t)$ *and sum over* j *to get*

$$\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega \Big(|u_t^k(x,t)|^2 dx + |\nabla u^k(x,t)|^2\Big) dx\Big] + a\int_\Omega |u_t^k(x,t)|^{m(x)} dx = \int_\Omega f(x,t) u_t^k(x,t) dx.$$

Integration over $(0, t)$ *gives*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big(|u_t^k(x,t)|^2 dx + |\nabla u^k(x,t)|^2\Big) dx + a\int_0^t \int_\Omega |u_t^k(x,s)|^{m(x)} dx ds \\
&= \frac{1}{2}\int_\Omega \big(|u_1^k|^2 + |\nabla u_0^k|^2\big) dx + \int_0^t \int_\Omega f(x,s) u_t^k(x,s) dx ds \\
&\leq \frac{1}{2}\int_\Omega \big(u_1^2 + |\nabla u_0|^2\big) dx + \varepsilon \int_0^T \int_\Omega |u_t^k|^2 dx ds + c_\varepsilon \int_0^T \int_\Omega f^2 dx ds \\
&\leq C_\varepsilon + \varepsilon \sup_{(0,t_k)} \int_\Omega |u_t^k(x,t)|^2 dx,\ \ \forall t \in [0, t_k).
\end{aligned} \tag{3.11}$$

So, we have

$$\frac{1}{2}\sup_{(0,t_k)}\int_\Omega |u_t^k(x,t)|^2dx+\frac{1}{2}\sup_{(0,t_k)}\int_\Omega |\nabla u^k(x,t)|^2dx+a\int_0^{t_k}\int_\Omega |u_t^k(x,s)|^{m(x)}dxds$$
$$\le C_\varepsilon+\varepsilon\sup_{(0,t_k)}\int_\Omega |u_t^k(x,t)|^2dx \tag{3.12}$$

Choosing $\varepsilon=\frac{1}{4}$, *we arrive at*

$$\sup_{(0,t_k)}\int_\Omega |u_t^k(x,t)|^2dx+\sup_{(0,t_k)}\int_\Omega |\nabla u^k(x,t)|^2dx+a\int_0^{t_k}\int_\Omega |u_t^k(x,s)|^{m(x)}dxds\le C$$

Thus, the solution can be extended to $[0,T)$ *and, in addition, we have*

$$(u^k) \text{ is a bounded sequence in } L^\infty\big((0,T),H_0^1(\Omega)\big)$$

$$(u_t^k) \text{ is a bounded sequence in } L^\infty\big((0,T),L^2(\Omega)\big)\cap L^{m(\cdot)}\big(\Omega\times(0,T)\big).$$

Therefore we can extract a subsequence (u^ℓ) *such that*

$$u^\ell\to u \text{ weakly}* \text{ in } L^\infty\big((0,T),H_0^1(\Omega)\big)$$

$$u_t^\ell\to u_t \text{ weakly}* \text{ in } L^\infty\big((0,T),L^2(\Omega)\big) \text{ and weakly in } L^{m(\cdot)}\big(\Omega\times(0,T)\big).$$

We can conclude by Lion's Lemma [69] that $u\in C\big([0,T],L^2(\Omega)\big)$ *so that* $u(x,0)$ *has a meaning. Since* (u_t^ℓ) *is bounded in* $L^{m(\cdot)}\big(\Omega\times(0,T)\big)$ *then* $\big(|u_t^\ell|^{m(x)-2}u_t^\ell\big)$ *is* *bounded in* $L^{\frac{m(\cdot)}{m(\cdot)-1}}\big(\Omega \times (0,T)\big)$*; hence, up to a subsequence,*

$$|u_t^\ell|^{m(\cdot)-2}u_t^\ell \to \psi \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}\big(\Omega \times (0,T)\big).$$

We have to show that $\psi = |u_t|^{m(\cdot)-2}u_t$*. In (3.9), we use* u^ℓ *instead of* u^k *and integrate over* $(0,t)$ *to obtain*

$$\int_\Omega u_t^\ell v_j - \int_\Omega u_1^\ell v_j + \int_0^t\int_\Omega \nabla u^\ell \cdot \nabla v_j + a\int_0^t\int_\Omega |u_t^\ell|^{m(x)-2}u_t^\ell v_j = \int_0^t\int_\Omega f v_j,\ \forall j < \ell.$$

As ℓ *goes to* $+\infty$*, we easily check that*

$$\int_\Omega u_t v_j - \int_\Omega u_1 v_j + \int_0^t\int_\Omega \nabla u \cdot \nabla v_j + a\int_0^t\int_\Omega \psi v_j = \int_0^t\int_\Omega f v_j,\ \forall j \geq 1.$$

Consequently,

$$\int_\Omega u_t v - \int_\Omega u_1 v + \int_0^t\int_\Omega \nabla u \cdot \nabla v + a\int_0^t\int_\Omega \psi v = \int_0^t\int_\Omega f v,\ \forall v \in H_0^1(\Omega).$$

All terms define absolute continuous functions; so we get, for a.e $t \in [0,T]$*,*

$$\frac{d}{dt}\int_\Omega u_t v + \int_\Omega \big(\nabla u \cdot \nabla v + a\psi v\big) = \int_\Omega f v,\ \forall v \in H_0^1(\Omega). \tag{3.13}$$

This implies that

$$u_{tt} - \Delta u + \psi = f,\ \text{ in } D'\big(\Omega \times (0,T)\big). \tag{3.14}$$

For simplicity, let $A(v) = |v|^{m(x)-2}v$ and define

$$X^\ell = \int_0^T \int_\Omega \big(A(u_t^\ell) - A(v)\big)(u_t^\ell - v)dt \geq 0, \quad \forall v \in L^{m(\cdot)}\big((0,T)H_0^1(\Omega)\big).$$

So, by using (3.11) and replacing u^k by u^ℓ, we get

$$\begin{aligned} X^\ell = &\int_0^T \int_\Omega f u_t^\ell + \frac{1}{2}\int_\Omega \big(|u_1^\ell|^2 + |\nabla u_0^\ell|^2\big) - \frac{1}{2}\int_\Omega |u_t^\ell(x,T)|^2 \\ &- \frac{1}{2}\int_\Omega |\nabla u^\ell(x,T)|^2 - \int_0^T \int_\Omega A(u_t^\ell)v - \int_0^T \int_\Omega A(v)(u_t^\ell - v). \end{aligned} \tag{3.15}$$

Taking $\ell \to \infty$, we obtain

$$\begin{aligned} 0 \leq \limsup_\ell X^\ell \leq &\int_0^T \int_\Omega f u_t + \frac{1}{2}\int_\Omega \big(u_1^2 + |\nabla u_0|^2\big) - \frac{1}{2}\int_\Omega |u_t(x,T)|^2 \\ &- \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 - \int_0^T \int_\Omega \psi v - \int_0^T \int_\Omega A(v)(u_t - v). \end{aligned} \tag{3.16}$$

Replacing v by u_t in (3.13) and integrating over $(0,T)$, we arrive at

$$\begin{aligned} \int_0^T \int_\Omega f u_t = &\frac{1}{2}\int_\Omega |u_t(x,T)|^2 - \frac{1}{2}\int_\Omega u_1^2 + \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 \\ &- \frac{1}{2}\int_\Omega |\nabla u_0|^2 + \int_0^T \int_\Omega \psi u_t. \end{aligned} \tag{3.17}$$

Addition of (3.16) and (3.17) yields

$$0 \leq \limsup_\ell X^\ell \leq \int_0^T \int_\Omega \psi u_t - \int_0^T \int_\Omega \psi v - \int_0^T \int_\Omega A(v)(u_t - v).$$

That is,

$$\int_0^T \int_\Omega \big(\psi - A(v)\big)(u_t - v)dt \geq 0, \ \ \forall v \in L^{m(\cdot)}\big((0,T)H_0^1(\Omega)\big).$$

Hence,

$$\int_0^T \int_\Omega \big(\psi - A(v)\big)(u_t - v)dt \geq 0, \ \ \forall v \in L^{m(\cdot)}\big(\Omega \times (0,T)\big),$$

by density of $H_0^1(\Omega)$ *in* $L^{m(\cdot)}(\Omega)$ *(Lemma 2.5).*

Let $v = \lambda w + u_t$, $w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big)$. *So, we get*

$$-\lambda \int_0^T \int_\Omega \big(\psi - A(\lambda w + u_t)\big)w \geq 0, \ \ \forall \lambda \neq 0, \ \forall w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big).$$

For $\lambda > 0$, *we have*

$$\int_0^T \int_\Omega \big(\psi - A(\lambda w + u_t)\big)w \leq 0, \ \ \forall w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big).$$

As $\lambda \to 0$ *and using the continuity of* A *with respect to* λ, *we get*

$$\int_0^T \int_\Omega \big(\psi - A(u_t)\big)w \leq 0, \ \ \forall w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big).$$

Similarly, for $\lambda < 0$, *we get*

$$\int_0^T \int_\Omega \big(\psi - A(u_t)\big)w \geq 0, \ \ \forall w \in L^{m(\cdot)}\big(\Omega \times (0,T)\big).$$

This implies that $\psi = A(u_t)$. *Hence (3.13) becomes*

$$\int_\Omega \big(u_{tt}v + \nabla u \cdot \nabla v + a|u_t|^{m(x)-2}u_t v\big) = \int_\Omega fv, \ \forall v \in L^{m(\cdot)}\big((0,T) \times H_0^1(\Omega)\big),$$

which gives

$$u_{tt} - \Delta u + a|u_t|^{m(x)-2}u_t = f, \ \textit{in } D'\big(\Omega \times (0,T)\big).$$

To handle the initial conditions, we note that

$$\begin{aligned} &u^l \rightharpoonup u \ \textit{weakly} * \textit{ in } L^\infty((0,T), H_0^1(\Omega)) \\ &u_t^l \rightharpoonup u_t \ \textit{weakly} * \textit{ in } L^\infty((0,T), L^2(\Omega)). \end{aligned} \tag{3.18}$$

Thus, using Lions' Lemma [69], we obtain

$$u^l \to u \quad \textit{in } C([0,T], L^2(\Omega)). \tag{3.19}$$

Therefore, $u^l(x,0)$ *makes sense and* $u^l(x,0) \to u(x,0)$ *in* $L^2(\Omega)$. *Also we have, by construction,*

$$u^l(x,0) = u_0^l(x) \to u_0(x) \quad \textit{in } H_0^1(\Omega).$$

Hence

$$u(x,0) = u_0(x). \tag{3.20}$$

As in [71], let $\phi \in C_0^\infty(0,T)$ *and replace* (u^k) *by* (u^l) *to obtain from (3.9) and for* *any $j \leq l$ that*

$$
\begin{aligned}
&-\int_0^T \int_\Omega u_t^l(x,t) v_j(x) \phi'(t) dx dt \\
&= -\int_0^T \int_\Omega \nabla u^\ell(x,t) \nabla v_j(x) \phi(t) dx dt - a \int_0^T \int_\Omega |u_t^\ell(x,t)|^{m(x)-2} u_t^\ell(x,t) v_j(x) \phi(t) dx dt \\
&\quad + \int_0^T \int_\Omega f(x,t) v_j(x) \phi(t) dx dt.
\end{aligned} \tag{3.21}
$$

As $l \to +\infty$, we obtain that

$$
\begin{aligned}
&-\int_0^T \int_\Omega u_t(x,t) v_j(x) \phi'(t) dx dt \\
&= -\int_0^T \int_\Omega \nabla u(x,t) \nabla v_j(x) \phi(t) dx dt - a \int_0^T \int_\Omega |u_t(x,t)|^{m(x)-2} u_t(x,t) v_j(x) \phi(t) dx dt \\
&\quad + \int_0^T \int_\Omega f(x,t) v_j(x) \phi(t) dx dt,
\end{aligned} \tag{3.22}
$$

for all $j \geq 1$. This implies

$$
\begin{aligned}
&-\int_0^T \int_\Omega u_t(x,t) v(x) \phi'(t) dx dt \\
&= \int_0^T \int_\Omega \left[\Delta u - a|u_t(x,t)|^{m(x)-2} u_t(x,t) + f(x,t)\right] v(x) \phi(t) dx dt,
\end{aligned} \tag{3.23}
$$

for all $v \in H_0^1(\Omega)$. This means $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega))$ and u solves the equation

$$u_{tt} - \Delta u + a|u_t|^{m(\cdot)-2} u_t = f. \tag{3.24}$$

Thus, $u_t \in L^\infty([0,T), L^2(\Omega))$, $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega))$. Consequently,

$$u_t \in C([0,T), H^{-1}(\Omega)). \tag{3.25}$$

So, $u_t^l(x,0)$ makes sense (see [71, p.116]). It follows that

$$u_t^l(x,0) \to u_t(x,0) \quad in\ H^{-1}(\Omega).$$

But

$$u_t^l(x,0) = u_1^l(x) \to u_1(x) \quad in\ L^2(\Omega).$$

Hence

$$u_t(x,0) = u_1(x). \tag{3.26}$$

This ends the proof of Proposition 3.1 ∎

Now, we give the wellposedness result of our problem. First we need the following lemma

Lemma 3.1 *For a.e $x \in \Omega$ and $p(\cdot)$ satisfying*

$$2 < p_1 \leq p(x) \leq p_2 < +\infty,$$

the function $g(s) = b|s|^{p(x)-2}s$ is differentiable and $|g'(s)| = |b||p(x)-1||s|^{p(x)-2}$.

Theorem 3.1 *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given and $m, p \in \mathcal{C}(\overline{\Omega})$. Under the assumptions $(A2), (A3)$, the problem (3.1) has a unique local solution*

$$\begin{gathered} u \in L^\infty\big((0,T), H_0^1(\Omega)\big), \quad u_t \in L^\infty\big((0,T), L^2(\Omega)\big) \cap L^{m(\cdot)}\big(\Omega \times (0,T)\big), \\ u_{tt} \in L^2\big((0,T), H^{-1}(\Omega)\big). \end{gathered} \tag{3.27}$$

Proof. *Let* $v \in L^\infty\big((0,T), H_0^1(\Omega)\big)$*. Then*

$$\|g(v)\|_2^2 = |b|^2 \int_\Omega |v|^{2(p(x)-1)}dx \le \quad |b|^2\Big[\int_\Omega |v|^{2(p_2-1)}dx + \int_\Omega |v|^{2(p_1-1)}dx\Big] < \quad +\infty,$$

since

$$2(p_1-1) \le 2(p_2-1) \le \frac{2n}{n-2}.$$

So, in this case,

$$g(v) \in L^\infty\big((0,T), L^2(\Omega)\big) \subset L^2\big(\Omega\times(0,T)\big).$$

Therefore, for each $v \in L^\infty\big((0,T), H_0^1(\Omega)\big)$*, there exists a unique*

$$u \in L^\infty\big((0,T), H_0^1(\Omega)\big), \;\; u_t \in L^\infty\big((0,T), L^2(\Omega)\big) \cap L^{m(\cdot)}\big(\Omega\times(0,T)\big)$$

satisfying the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + a u_t|u_t|^{m(\cdot)-2} = g(v), & \text{in } \Omega\times(0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{3.28}$$

We define a map $G : X_T \to X_T$ *by* $G(v) = u$*, where*

$$X_T = \big\{w \in L^\infty\big((0,T), H_0^1(\Omega)\big) \;/\; w_t \in L^\infty\big((0,T), L^2(\Omega)\big)\big\}.$$

X_T is Banach space with respect to the norm

$$\|w\|_{X_T} = \|w\|_{L^\infty\left((0,T),H_0^1(\Omega)\right)} + \|w_t\|_{L^\infty\left((0,T),L^2(\Omega)\right)}.$$

Multiply the equation in (3.28) by u_t and integrate over $\Omega \times (0,t)$, to get

$$\begin{aligned}\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + a\int_0^t\int_\Omega |u_t|^{m(x)}dx &= \frac{1}{2}\int_\Omega u_1^2 + \frac{1}{2}\int_\Omega |\nabla u_0|^2 \\ &+ b\int_0^t\int_\Omega |v|^{p(x)-2}vu_t dx.\end{aligned} \tag{3.29}$$

Using Young's inequality, we have

$$\begin{aligned}\Big|\int_\Omega |v|^{p(x)-2}vu_t\Big| &\le \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{4}{\varepsilon}\int_\Omega |v|^{2p(x)-2}dx \\ &\le \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{4}{\varepsilon}\Big[\int_\Omega |v|^{2p_2-2}dx + \int_\Omega |v|^{2p_1-2}dx\Big] \\ &\le \frac{\varepsilon}{4}\int_\Omega u_t^2 dx + \frac{c_e}{\varepsilon}\Big[\|\nabla v\|_2^{2p_2-2} + \|\nabla v\|_2^{2p_1-2}\Big].\end{aligned}$$

Thus (3.29) becomes

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le \lambda_0 + \frac{|b|\varepsilon T}{4}\sup_{(0,T)}\int_\Omega u_t^2 + \frac{|b|c_e}{\varepsilon}\Big[\int_0^T \|\nabla v\|_2^{2p_2-2} + \int_0^T \|\nabla v\|_2^{2p_1-2}\Big];$$

hence we have

$$\frac{1}{2}\sup_{(0,T)}\int_\Omega u_t^2 + \frac{1}{2}\sup_{(0,T)}\int_\Omega |\nabla u|^2 \le 2\lambda_0 + \frac{|b|\varepsilon T}{2}\sup_{(0,T)}\int_\Omega u_t^2 + Tc_\varepsilon\Big[\|v\|_{X_T}^{2p_2-2} + \|v\|_{X_T}^{2p_1-2}\Big],$$

where

$$\lambda_0 = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 \ \ \textit{and} \ \ c_e \ \ \textit{is the embedding constant.}$$

Choosing ε *such that* $\frac{|b|\varepsilon T}{2} = \frac{1}{4}$, *we get*

$$\|u\|_{X_T}^2 \le \lambda + T\beta\Big[\|v\|_{X_T}^{2p_2-2} + \|v\|_{X_T}^{2p_1-2}\Big].$$

Suppose that $\|v\|_{X_T} \le M$, *for some* M *large. Then*

$$\|u\|_{X_T}^2 \le \lambda + T\beta M^{2p_2-2} \le M^2,$$

if

$$M^2 > \lambda \quad \textit{and} \quad T \le T_0 < \frac{M^2-\lambda}{\beta M^{2p_2-2}}.$$

We conclude that $G : B \to B$, *where*

$$B = \big\{w \in L^\infty\big((0,T), H_0^1(\Omega)\big)\ ,\ w_t \in L^\infty\big((0,T), L^2(\Omega)\big) \ \textit{such that} \ \|w\|_{X_{T_0}} \le M\}.$$

Next, we show that, for T_0 *(even smaller),* G *is a contraction. For this purpose,* *let* $u_1 = G(v_1)$ *and* $u_2 = G(v_2)$ *and set* $u = u_1 - u_2$ *then* u *satisfies*

$$\begin{cases} u_{tt} - \Delta u + a\Big[u_{1t}|u_{1t}|^{m(\cdot)-2} - u_{2t}|u_{2t}|^{m(\cdot)-2}\Big] \\ = b\Big[|v_1|^{p(x)-2}v_1 - |v_2|^{p(x)-2}v_2\Big], & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{3.30}$$

Multiplication by u_t *and integration over* $\Omega \times (0,t)$ *yield*

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + a\int_0^t\int_\Omega \Big[|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\Big](u_{1t} - u_{2t}) \\ &= b\int_0^t\int_\Omega \big(g(v_1) - g(v_2)\big)u_t dxds; \end{aligned} \tag{3.31}$$

hence we have

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \leq b\int_0^t\int_\Omega \big(g(v_1) - g(v_2)\big)u_t dxds. \tag{3.32}$$

Now, we evaluate

$$I = \int_\Omega \big|g(v_1) - g(v_2)\big||u_t| = \int_\Omega |g'(\xi)||v||u_t|,$$

where

$$v = v_1 - v_2 \text{ and } \xi = \alpha v_1 + (1-\alpha)v_2,\ 0 \leq \alpha \leq 1.$$

Young's inequality implies

$$\begin{aligned}
I &\leq \frac{\delta}{2}\int_\Omega u_t^2 + \frac{2}{\delta}\int_\Omega |g'(\xi)|^2|v|^2 \\
&\leq \frac{\delta}{2}\int_\Omega u_t^2 + \frac{2a^2(p_2-1)^2}{\delta}\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{2(p(x)-2)}|v|^2 \\
&\leq \frac{\delta}{2}\int_\Omega u_t^2 + c_\delta\Big(\int_\Omega |v|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\Big[\Big(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(p_2-2)}\Big)^{\frac{2}{n}} \\
&+ \Big(\int_\Omega |\alpha v_1 + (1-\alpha)v_2|^{n(p_1-2)}\Big)^{\frac{2}{n}}\Big]
\end{aligned}$$

By recalling (3.4), we arrive at

$$\begin{aligned}
I &\leq \frac{\delta}{2}\int_\Omega u_t^2 + c_\delta c_e\|\nabla v\|_2^2\big[\|\nabla v_1\|_2^{2(p_2-2)} + \|\nabla v_1\|_2^{2(p_1-2)} + \|\nabla v_2\|_2^{2(p_2-2)} + \|\nabla v_2\|_2^{2(p_1-2)}\big] \\
&\leq \frac{\delta}{2}\int_\Omega u_t^2 + 4c_\delta c_e M^{2(p_2-2)}\|\nabla v\|_2^2
\end{aligned}$$

Therefore, (3.32) takes the form

$$\frac{1}{2}\|u\|_{X_T}^2 \leq \frac{\delta}{2}T_0 b\|u\|_{X_T}^2 + C_\delta M^{2(p_2-2)}T_0 b\|v\|_{X_T}^2.$$

Choosing δ small enough, we arrive at

$$\|u\|_{X_T}^2 \leq 4C_\delta M^{2(p_2-2)}T_0 b\|v\|_{X_T}^2 = \gamma T_0\|v\|_{X_T}^2.$$

Taking T_0 small enough, we get

$$\|u\|_{X_T}^2 \leq d\|v\|_{X_T}^2, \ \ \textit{for} \ \ 0 < d < 1.$$

Thus G is a contraction. The Banach fixed theorem implies the existence of a unique $u \in B$ satisfying $G(u) = u$. Thus, u is a local solution of (3.8).

Uniqueness. *Suppose we have two solutions u and v. Then $w = u - v$ satisfies*

$$\begin{cases} w_{tt} - \Delta w + au_t|u_t|^{m(\cdot)-2} - av_t|v_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2} - bv|v|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = w_t(x,0) = 0, & \text{in } \Omega, \end{cases}$$

Multiply by w_t and integrate over $\Omega \times (0,t)$ to obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega w_t^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + a\int_0^t \int_\Omega \big(u_t|u_t|^{m(x)-2} - v_t|v_t|^{m(x)-2}\big)\big(u_t - v_t\big)dx \\ &= b\int_0^t \int_\Omega \big(u|u|^{p(x)-2} - v|v|^{p(x)-2}\big)w_t dx. \end{aligned} \tag{3.33}$$

This implies

$$\frac{1}{2}\int_\Omega w_t^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 \leq b\int_0^t \int_\Omega \big(u|u|^{p(x)-2} - v|v|^{p(x)-2}\big)w_t dx.$$

By repeating the same estimates as in above, we arrive at

$$\int_\Omega \big(w_t^2 + |\nabla w|^2\big) \leq C\int_0^t \int_\Omega \big(w_t^2(x,s) + |\nabla w(x,s)|^2\big)dxds$$

Gronwall's inequality yields

$$\int_\Omega \big(w_t^2 + |\nabla w|^2\big) = 0.$$

Thus, $w \equiv 0$. This shows the uniqueness. The proof of Theorem 3.1 is completed.

∎

3.3 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our blow-up result.

Lemma 3.2 *Suppose the conditions of Corollary 2.6 hold. Then there exists a positive $C > 1$, depending on Ω only, such that*

$$\varrho^{\frac{s}{p_1}}(u) \leq C\big(\|\nabla u\|_2^2 + \varrho(u)\big), \tag{3.34}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.

Proof. If $\varrho(u) > 1$, then $\varrho^{\frac{s}{p_1}}(u) \leq \varrho(u) \leq C\big(\|\nabla u\|_2^2 + \varrho(u)\big)$, where $C > 1$. If $\varrho(u) \leq 1$ then, by Lemma 2.3, $\|u\|_{p(\cdot)} \leq 1$. Then, Corollary 2.6 and Lemma 2.4 imply

$$\varrho^{\frac{s}{p_1}}(u) \leq \varrho^{\frac{2}{p_1}}(u) \leq \Big[\max\big\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\big\}\Big]^{\frac{2}{p_1}} = \|u\|_{p(\cdot)}^2 \leq C\|\nabla u\|_2^2.$$

Therefore (3.34) follows. ∎

As a special case, we have

Corollary 3.2 *Let the assumptions of Lemma 3.2 hold. Then we have*

$$\|u\|_{p_1}^s \leq C\big(\|\nabla u\|_2^2 + \|u\|_{p_1}^{p_1}\big), \tag{3.35}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

We set

$$H(t) := -E(t)$$

and use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (3.6) and (3.34), we have

Corollary 3.3 *Let the assumptions of Lemma 3.2 hold. Then we have*

$$\varrho^{\frac{s}{p_1}}(u) \leq C\big(|H(t)| + \|u_t\|_2^2 + \varrho(u)\big), \tag{3.36}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

As a special case, we have

Corollary 3.4 *Let the assumptions of Lemma 3.2 hold. Then we have*

$$\|u\|_{p_1}^s \leq C\big(|H(t)| + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big), \tag{3.37}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

Lemma 3.3 *Let the assumptions of Lemma 3.2 hold and let u be the solution of* *problem (3.1). Then,*

$$\varrho(u) \geq C\|u\|_{p_1}^{p_1}. \tag{3.38}$$

Proof.

$$\varrho(u) = \int_\Omega |u|^{p(x)}dx = \int_{\Omega_+} |u|^{p(x)}dx + \int_{\Omega_-} |u|^{p(x)}dx,$$

where

$$\Omega_+ = \big\{x \in \Omega \;/\; |u(x,t)| \geq 1\big\} \quad \text{and} \quad \Omega_- = \big\{x \in \Omega \;/\; |u(x,t)| < 1\big\}.$$

So we get

$$\varrho(u) \geq \int_{\Omega+} |u|^{p_1} + \int_{\Omega_-} |u|^{p_2} \geq \int_{\Omega+} |u|^{p_1} + c_1\Big(\int_{\Omega-} |u|^{p_1}\Big)^{\frac{p_2}{p_1}}.$$

This implies that

$$c_2\big(\varrho(u)\big)^{\frac{p_1}{p_2}} \geq \int_{\Omega-} |u|^{p_1} \;\text{ and }\; \varrho(u) \geq \int_{\Omega+} |u|^{p_1}.$$

This yields, by addition,

$$c_2\big(\varrho(u)\big)^{\frac{p_1}{p_2}} + \varrho(u) \geq \|u\|_{p_1}^{p_1}. \tag{3.39}$$

Since

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\varrho(u),$$

then (3.39) leads to

$$\varrho(u)\Big[1+c_2\Big(\frac{p_1}{b}H(0)\Big)^{\frac{p_1}{p_2}-1}\Big] \geq \|u\|_{p_1}^{p_1}.$$

Hence, (3.38) follows. ∎

Lemma 3.4 *Assume that* (3.5) *holds and let u be the solution of problem (3.1). Then,*

$$\int_\Omega |u|^{m(x)}dx \leq C\Big(\big(\varrho(u)\big)^{\frac{m_1}{p_1}} + \big(\varrho(u)\big)^{\frac{m_2}{p_1}}\Big). \tag{3.40}$$

Proof. Recalling the definitions of Ω_+ and Ω_-, and using Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |u|^{m(x)}dx &\leq \int_{\Omega_-} |u|^{m_1}dx + \int_{\Omega_+} |u|^{m_2}dx \\
&\leq C\Big[\Big(\int_{\Omega_-} |u|^{p_1}dx\Big)^{m_1/p_1} + \Big(\int_{\Omega_+} |u|^{p_1}dx\Big)^{m_2/p_1}\Big] \\
&\leq C\big(\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2}\big) \\
&\leq C\Big(\big(\varrho(u)\big)^{\frac{m_1}{p_1}} + \big(\varrho(u)\big)^{\frac{m_2}{p_1}}\Big),
\end{aligned}$$

by Lemma 3.3. ∎

3.4 The Main Blow-Up Result

In this section we state and prove our main blow up result.

Theorem 3.5 *Let the conditions of Theorem 3.1 be fulfilled. Assume further that*

(A4) holds and

$$E(0) < 0. \tag{3.41}$$

Then the solution (3.27) blows up in finite time.

Proof. We multiply equation (3.1) by u_t and integrate over Ω to get

$$E'(t) = -a \int_\Omega |u_t(x,t)|^{m(x)} dx, \tag{3.42}$$

for almost every t in $[0,T)$ since $E(t)$ is absolutely continuous (see [34]); hence $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1} \varrho(u), \tag{3.43}$$

for every t in $[0,T)$, by virtue of (3.41). We then define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t) dx, \tag{3.44}$$

for ε small to be chosen later and

$$0 < \alpha \leq \min\left\{ \frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)} \right\}. \tag{3.45}$$

By taking the derivative of (3.44) and using equation (3.1), we obtain

$$L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right] + \varepsilon b \int_\Omega |u|^{p(x)} - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2}. \tag{3.46}$$

Add and subtract $\varepsilon(1-\eta)p_1 H(t)$, for $0 < \eta < 1$, in the right-hand side of (3.46), to arrive at

$$
\begin{aligned}
Ł'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1 H(t) + \varepsilon b\eta \int_\Omega |u|^{p(x)} \\
&\quad + \varepsilon\Big(\frac{(1-\eta)p_1}{2}+1\Big)\|u_t\|_2^2 + \varepsilon\Big(\frac{(1-\eta)p_1}{2}-1\Big)\|\nabla u\|_2^2 - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2} dx.
\end{aligned}
\tag{3.47}
$$

For η small enough, we see that

$$
L'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\beta\big[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\varrho(u)\big] - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2} dx,
\tag{3.48}
$$

where

$$
\beta = \min\Big\{(1-\eta)p_1, b\eta, \frac{(1-\eta)p_1}{2}+1, \frac{(1-\eta)p_1}{2}-1\Big\} > 0.
$$

Now, by using Young's inequality, we estimate the last term in (3.48) as follows

$$
\int_\Omega |u_t|^{m(x)-1}|u| dx \leq \frac{1}{m_1}\int_\Omega \delta^{m(x)}|u|^{m(x)} dx + \frac{m_2-1}{m_2}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)} dx, \quad \forall \delta > 0.
\tag{3.49}
$$

Therefore by taking δ so that

$$
\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),
$$

for a large constant k to be specified later, and substituting in (3.49) we obtain

$$
\int_\Omega |u_t|^{m(x)-1}|u| dx \leq \frac{1}{m_1}\int_\Omega k^{1-m(x)}|u|^{m(x)} H^{\alpha(m(x)-1)}(t) dx + \frac{(m_2-1)k}{am_2} H^{-\alpha}(t)H'(t).
\tag{3.50}
$$

Combination of (3.48) and (3.50) gives

$$\text{Ł}'(t) \geq \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k\Big]H^{-\alpha}(t)H'(t) + \varepsilon\beta\big[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \varrho(u)\big]$$
$$- \varepsilon\frac{k^{1-m_1}a}{m_1}\int_\Omega H^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx. \tag{3.51}$$

By using Lemma 3.4 and (3.43), we estimate the last term of (3.51) as follows

$$\begin{aligned}\int_\Omega H^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx &= \int_\Omega \big[H(t)/H(0)\big]^{\alpha(m(x)-1)}[H(0)]^{\alpha(m(x)-1)}|u|^{m(x)}dx \\ &\leq \int_\Omega [H(0)]^{\alpha(m(x)-m_2)}[H(t)]^{\alpha(m_2-1)}|u|^{m(x)}dx \\ &\leq C_1[H(t)]^{\alpha(m_2-1)}\int_\Omega |u|^{m(x)}dx \\ &\leq C_1(b/p_1)^{\alpha(m_2-1)}(\varrho(u))^{\alpha(m_2-1)}\int_\Omega |u|^{m(x)}dx \\ &\leq C\Big[\big(\varrho(u)\big)^{\frac{m_1}{p_1}+\alpha(m_2-1)} + \big(\varrho(u)\big)^{\frac{m_2}{p_1}+\alpha(m_2-1)}\Big].\end{aligned} \tag{3.52}$$

We then use Lemma 3.2 and (3.45), for

$$s = m_2 + \alpha p_1(m_2-1) \leq p_1 \;\text{ and }\; s = m_1 + \alpha p_1(m_2-1) \leq p_1,$$

to deduce, from (3.52),

$$\int_\Omega H^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx \leq C\Big(\|\nabla u\|_2^2 + \varrho(u)\Big). \tag{3.53}$$

Combining (3.51) and (3.53) yields

$$\begin{aligned}
Ł'(t) \geq & \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k\Big] H^{-\alpha}(t)H'(t) \\
& + \varepsilon\Big(\beta - \frac{k^{1-m_1}a}{m_1}C\Big)\big[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \varrho(u)\big].
\end{aligned} \tag{3.54}$$

At this point, we choose k large enough so that

$$\gamma = \beta - \frac{ak^{1-m_1}}{m_1}C > 0.$$

Once k is fixed (hence γ), we pick ε small enough so that

$$(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k \geq 0 \quad \text{and} \quad L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)dx > 0.$$

Therefore (3.54) takes the form

$$L'(t) \geq \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + \varrho(u)\big] \geq \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big], \tag{3.55}$$

by virtue of (3.38). Consequently we have

$$L(t) \geq L(0) > 0, \ \text{ for all } \ t \geq 0.$$

Next, we would like to show that

$$L'(t) \geq \Gamma L^{\frac{1}{1-\alpha}}(t), \ \text{ for all } \ t \geq 0, \tag{3.56}$$

where Γ is a positive constant depending only on $\varepsilon\gamma$ and C (the constant of Corollary 3.2). Once (3.56) is established, we obtain in a standard way the finite time blow up of $L(t)$. To prove (3.56), we first note that

$$\Big|\int_\Omega u u_t(x,t)dx\Big| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_{p_1}\|u_t\|_2,$$

which implies

$$\Big|\int_\Omega u u_t(x,t)dx\Big|^{\frac{1}{1-\alpha}} \leq C\|u\|_{p_1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Again Young's inequality gives

$$\Big|\int_\Omega u u_t(x,t)dx\Big|^{1/(1-\alpha)} \leq C\Big[\|u\|_{p_1}^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\Big], \tag{3.57}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta = 2(1-\alpha)$, to get $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p_1$ by (3.45). Therefore (3.57) becomes

$$\Big|\int_\Omega u u_t(x,t)dx\Big|^{1/(1-\alpha)} \leq C\Big[\|u\|_{p_1}^{s} + \|u_t\|_2^{2}\Big],$$

where $s = 2/(1-2\alpha) \leq p_1$. By using Corollary 3.4, we obtain

$$\Big|\int_\Omega u u_t(x,t)dx\Big|^{1/(1-\alpha)} \leq C\Big[H(t) + \|u\|_{p_1}^{p_1} + \|u_t\|_2^{2}\Big], \quad \text{for all } \ t \geq 0. \tag{3.58}$$

Finally, by noting that

$$
\begin{aligned}
L^{1/(1-\alpha)}(t) &= \left[H^{(1-\alpha)}(t) + \varepsilon \int_\Omega u u_t(x,t)dx\right]^{1/(1-\alpha)} \\
&\leq 2^{1/(1-\alpha)}\left[H(t) + \left|\int_\Omega u u_t\right|^{1/(1-\alpha)}\right]
\end{aligned}
$$

and combining it with (3.55) and (3.58), the inequality (3.56) is established. A simple integration of (3.56) over $(0,t)$ then yields

$$
L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}. \tag{3.59}
$$

Therefore (3.59) shows that $L(t)$ blows up in finite time

$$
T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}, \tag{3.60}
$$

where Γ and α are positive constant with $\alpha < 1$ and L is given by (3.44) above. This completes the proof. ▌

Remark 3.1 The estimate (3.60) shows that the larger $L(0)$ is, the quicker the blow up takes place.

3.5 Numerical Study

In this section, we present an application to illustrate numerically the blow-up result of Theorem 3.5. For this purpose, we numerically solve problem (3.1), for $n = 2$, where the domain Ω is taken to be the unit disk, and the parameters $a = 1$, $b = 1$, $u_1(x, y) = 0$, and $u_0(x, y) = k(1 - x^2 - y^2)$, where k will be chosen such that $E(0) < 0$. We take the exponent functions $m(x, y) = \lceil x \rceil^2 + 2.5$ and $p(x, y) = \lceil x \rceil^2 + 4$, which satisfy condition $(A1)$, where $\lceil \cdot \rceil$ denotes the greatest integer function. Note that condition $(A2)$ is required for $n \geq 3$, while here since $m, p > 2$, the conclusion of corollary 2.6 follows for $n = 1$ or $n = 2$.

3.5.1 Numerical Method

We first introduce a suitable numerical scheme to discretize problem (3.1) using finite differences for the time variable $t \in [0, T]$ and a finite element method for the space variable $(x, y) \in \Omega$. Comprehensive details about the finite difference and the finite element methods are available in [61], [85], [86]. We subdivide the time interval $[0, T]$ into N equal subintervals

$$[t_{n-1}, t_n], \quad t_n = n\tau, \quad n = 1, 2, ..., N + 1,$$

where τ is the time step. Let

$$U^n(x, y) := u(x, y, t_n),$$

and use the finite-difference formulas: the first-order backward difference for

$$\partial_t U^n(x,y) := \frac{U^n(x,y) - U^{n-1}(x,y)}{\tau},$$

and the second-order center difference for

$$\partial_{tt} U^n(x,y) := \frac{U^{n+1}(x,y) - 2U^n(x,y) + U^{n-1}(x,y)}{\tau^2}.$$

Then the time discrete problem of (3.1) reads: Given u_0 and u_1, find $\{U^2, U^3, ..., U^{n+1}\}$ such that

$$\begin{cases} -\Delta U^{n+1} + \frac{1}{\tau^2} U^{n+1} = \frac{2U^n - U^{n-1}}{\tau^2} + U^n |U^n|^{p(x,y)-2} - U^n |\partial_t U^n|^{m(x,y)-2}, & \text{in } \Omega \\ U^{n+1} = 0, & \text{on } \partial\Omega \\ U^0 = u_0(x,y), \quad U^1 = U^0 + \tau u_1(x,y) & \text{in } \Omega, \end{cases} \tag{3.61}$$

where the last equation is the discrete form of $u_t(x,y,0) = u_1(x,y)$, using the forward difference formula. Note also that the above problem is linear in U^{n+1}, which is achieved by using the history data U^n in the nonlinear terms.

Problem (3.61) is solved iteratively as for given regular U^{n-1} and U^n, the solution U^{n+1} satisfies the boundary-value problem:

$$\begin{cases} -\Delta U^{n+1} + \frac{1}{\tau^2} U^{n+1} = F(U^{n-1}, U^n), & \text{in } \Omega \\ U^{n+1} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.62}$$

where

$$F(U^{n-1}, U^n) := \frac{2U^n - U^{n-1}}{\tau^2} + U^n|U^n|^{p(x,y)-2} - U^n|\partial_t U^n|^{m(x,y)-2}.$$

We solve for U^{n+1} by using a finite element method as follows. Let Ω_h be a triangulation of Ω with a maximal diameter size of h and let $P_1(\Omega_h)$ be the linear Lagrangian subspace of $H_0^1(\Omega_h)$.

Now let $W \in P_1(\Omega_h)$ be any test function. Then, problem (3.62) has the following weak formulation:

$$\int_{\Omega_h} \left(\nabla U^{n+1} \cdot \nabla W + \frac{1}{\tau^2} U^{n+1} W \right) d\Omega_h = \int_{\Omega_h} F(U^{n-1}, U^n) W d\Omega_h. \tag{3.63}$$

The full-discrete problem (3.63) is elliptic and posses a unique solution $U^{n+1} \in P_1(\Omega_h)$ for every $n \geq 1$, provided that the history data U^{n-1}, U^n are regular enough. This follows from Lax-Milgram Lemma, see [61].

3.5.2 Numerical Results

In this subsection, we present and discuss the blow up results of the numerical scheme (3.63). The numerical results are obtained using the FreeFem++ software, which is an open source [39], in addition to Matlab.

We set the parameters as follows.

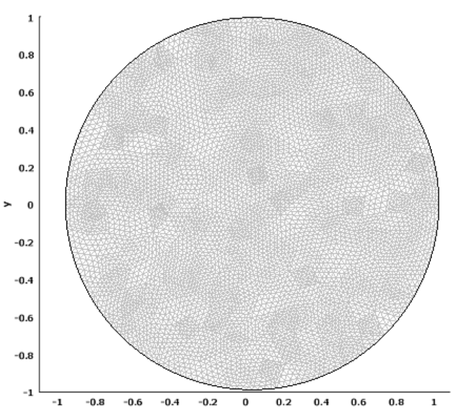
1. $k = 5$ makes the initial energy negative, $E(0) = -26.92$, as required.

2. The time step $\tau = 0.01$ is small enough to catch the below up behavior.

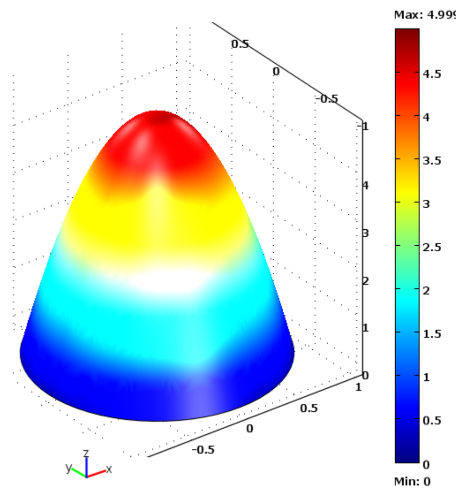
3. The triangulation Ω_h consists of 12096 triangles with 6145 degrees of freedoms.

Fig. 3.1 shows the mesh Ω_h, and the graphs of the initial data u_0 together the exponent functions m, p projected into $P_1(\Omega_h)$.

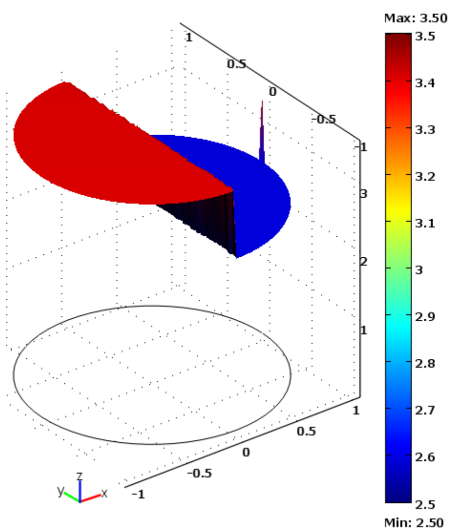
Fig. 3.2 and Fig. 3.3 present the solution U^n from iteration $n = 3$ ($t = 0.03$) to the final iteration $n = 43$ ($t = 0.43$) at which the blow-up occurs.



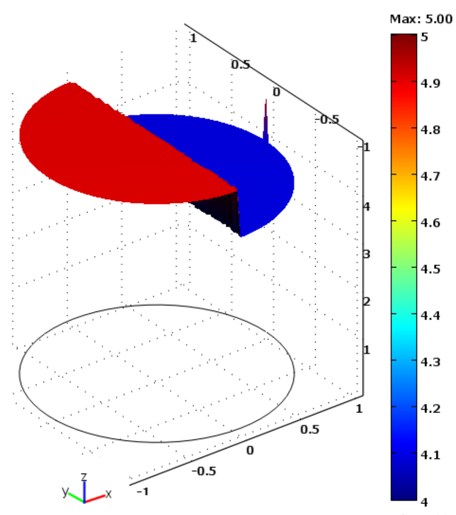
(a) The mesh Ω_h



(b) $u_0(x, y) = 5(1 - x^2 - y^2)$

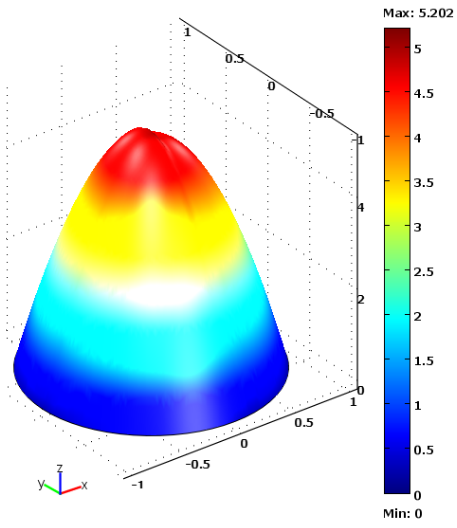


(c) $m(x, y) = 2.5 + \lceil x \rceil^2$

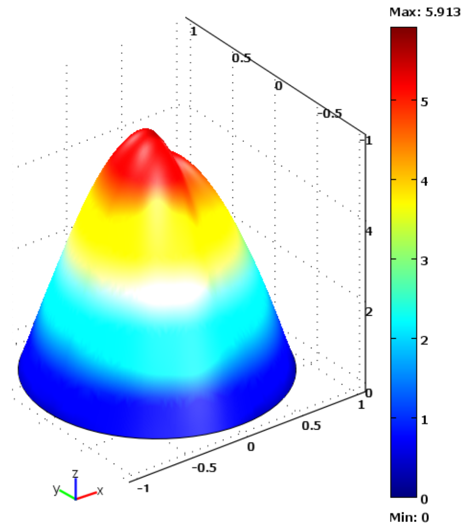


(d) $p(x, y) = 4 + \lceil x \rceil^2$

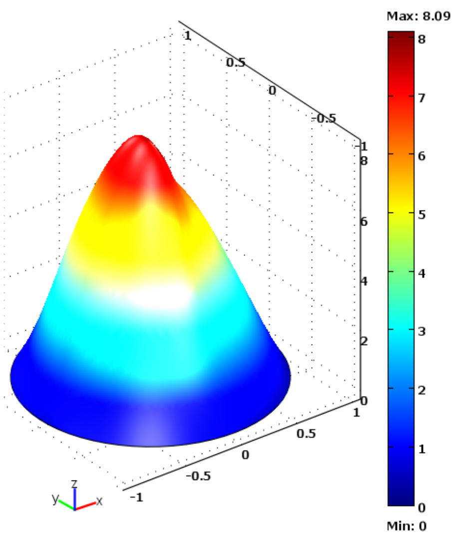
Figure 3.1: The graphs of mesh Ω_h, u_0, the exponent functions m and p.



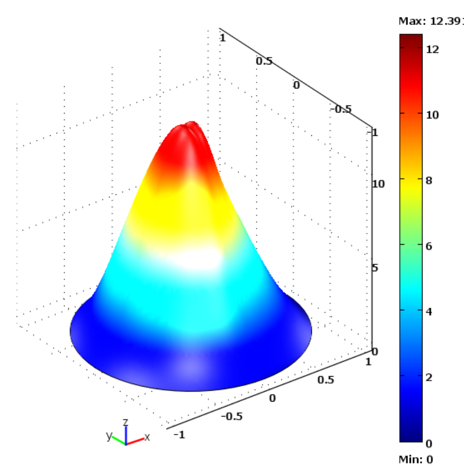
(a) U^3



(b) U^8

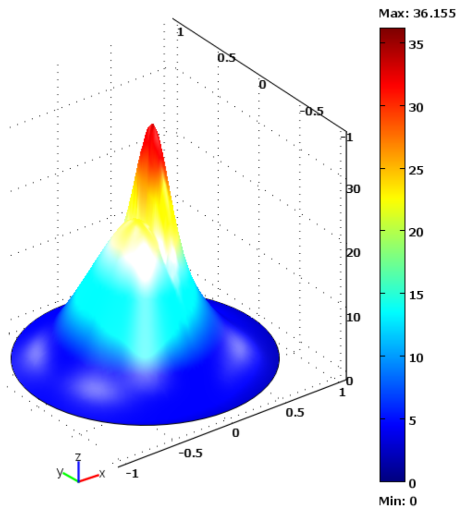


(c) U^{18}

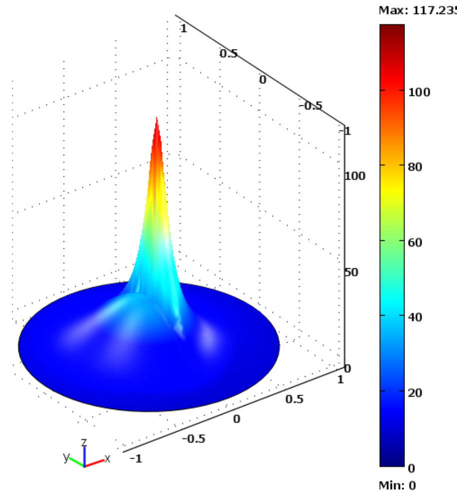


(d) U^{28}

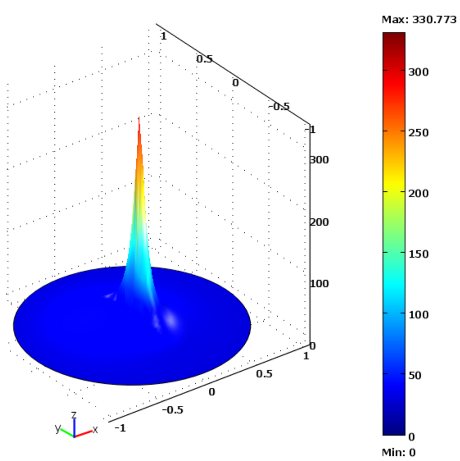
Figure 3.2: The solution U^n for iterations $n = 3, 8, 18, 28$.



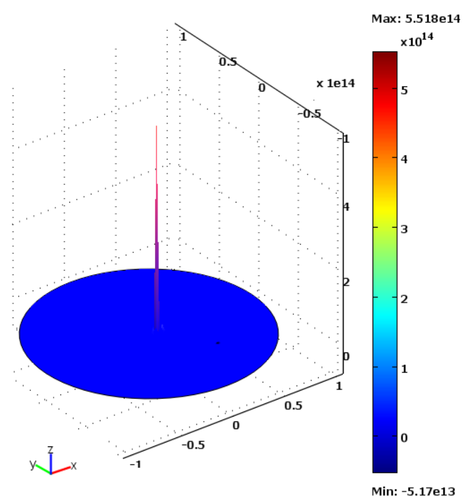
(a) U^{38}



(b) U^{41}



(c) U^{42}



(d) U^{43}

Figure 3.3: The solution U^n for iterations $n = 38, 41, 42, 43$.

Table 3.1: Numerical values of $\|U^n\|_\infty$ and the energy $E(t_n)$.

t_n	$\|U^n\|_\infty$	$E(t_n)$
0	4.99	-26.92
0.03	5.20	-27.72
0.08	5.91	-57.08
0.18	8.09	-421.53
0.28	12.39	-3576.77
0.38	36.15	-5.28E+4
0.41	117.23	-1.74E+5
0.42	330.77	-3.38E+6
0.43	5.51E+14	-2.77E+55

The table above (Table 3.1) lists numerical values of $\|U^n\|_\infty$ and the energy $E(t_n)$. It indicates the blowup of both the solution and the energy takes place at $t = 0.43$ as their magnitude orders drastically jumped high.

In conclusion, the above numerical application verifies and agrees with the blowup results of Theorem 3.5.

CHAPTER 4

BLOW UP IN SOLUTIONS OF A QUASILINEAR WAVE EQUATION WITH VARIABLE-EXPONENT NONLINEARITIES

In this chapter, we prove the finite time blow-up of solutions of the following quasilinear wave equation with variable exponents:

$$\begin{cases} u_{tt} - \operatorname{div}\big(|\nabla u|^{r(\cdot)-2}\nabla u\big) + au_t|u_t|^{m(\cdot)-2} = bu|u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $a, b > 0$ are constants, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and the exponents m, p and r are given measurable functions on Ω. In Section 4.1, we introduce some assumptions needed in this chapter. Some technical lemmas and the statement without proof of the wellposedness of our problem will be given in Section 4.2. In Section 4.3, we give a finite-time blowup result for the solutions with negative initial energy. In Section 4.4, we prove a finite-time blowup result for certain solutions with positive energy.

4.1 Assumptions

In this section, we present some materials needed in the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the variable-exponent Sobolev space $W_0^{1,r(\cdot)}(\Omega)$ with their norms. We assume the following hypotheses

$(A1)$ The exponents m, p and r are measurable functions such that either $m, p, r \in C(\overline{\Omega})$ or they satisfy the following log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \quad \text{with } |x-y| < \delta, \tag{4.2}$$

$A > 0, \quad 0 < \delta < 1.$

$(A2)$ m, p and r satisfy the following condition

$$2 \leq \max\{m_2, r_2\} < p_1 \leq p(x) \leq p_2 \leq r_*(x), \tag{4.3}$$

where

$$r_*(x) = \begin{cases} \frac{nr(x)}{\text{esssup}_{x\in\Omega}(n-r(x))} & \text{if } r_2 < n \\ \\ +\infty & \text{if } r_2 \geq n \end{cases}$$

and

$$\text{essinf}_{x\in\Omega}(r^*(x) - p(x)) > 0.$$

We introduce the energy associated to problem (4.1)

$$E(t) := \frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} dx - b\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx, \ t \geq 0. \tag{4.4}$$

Direct differentiation, using problem (4.1), leads to

$$E'(t) = -a\int_\Omega |u_t(x,t)|^{m(x)} dx, \text{ for a.e. } t \in [0,T). \tag{4.5}$$

4.2 The Well-posedness of the Problem

In this section we give the statement without proof of the wellposedness of our problem (4.1) as well as some lemmas which needed for the proof of our blow-up result.

Proposition 4.1 *Let $u_0 \in W_0^{1,r(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$ be given and $m, p, r \in \mathcal{C}(\overline{\Omega})$. Under condition $(A2)$, the problem (4.1) has a unique weak solution such that*

$$u \in L^\infty\Big((0,T), W_0^{1,r(\cdot)}(\Omega)\Big), \quad u_t \in L^\infty\Big((0,T), L^2(\Omega)\Big), \quad u_{tt} \in L^\infty\Big((0,T), W^{-1,r'(\cdot)}(\Omega)\Big),$$

where $\frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)} = 1.$

Remark 4.1 *The proof of this proposition can be established employing the Galerkin method as in the previous chapter. See also [11].*

Lemma 4.1 *Suppose the conditions of Lemma 2.8 hold. Then, there exists a constant $C > 1$, which depends on Ω only, such that*

$$\varrho_{p(\cdot)}^{\frac{s}{p_1}}(u) \leq C\big(\|\nabla u\|_{r(\cdot)}^{r_1} + \varrho_{p(\cdot)}(u)\big), \tag{4.6}$$

for any $u \in W_0^{1,r(\cdot)}(\Omega)$ and $r_1 \leq s \leq p_1$.

Proof. If $\varrho_{p(\cdot)}(u) > 1$, then $\varrho_{p(\cdot)}^{\frac{s}{p_1}}(u) \leq \varrho_{p(\cdot)}(u) \leq C\big(\|\nabla u\|_{r(\cdot)}^{r_1} + \varrho_{p(\cdot)}(u)\big)$. If $\varrho_{p(\cdot)}(u) \leq 1$ then, by Lemma 2.3, $\|u\|_{p(\cdot)} \leq 1$. Then, Lemma 2.8 and Lemma 2.4 imply

$$\varrho_{p(\cdot)}^{\frac{s}{p_1}}(u) \leq \varrho_{p(\cdot)}^{\frac{r_1}{p_1}}(u) \leq \Big[\max\big\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\big\}\Big]^{\frac{r_1}{p_1}} = \|u\|_{p(\cdot)}^{r_1} \leq C\|\nabla u\|_{r(\cdot)}^{r_1},$$

where $C > 1$. Therefore (4.6) follows. ∎

As a special case, we have the following

Corollary 4.1 *Under the assumptions of Lemma 4.1, we have*

$$\|u\|_{p_1}^{s} \leq C\big(\|\nabla u\|_{r(\cdot)}^{r_1} + \|u\|_{p_1}^{p_1}\big), \tag{4.7}$$

for any $u \in W_0^{1,r(\cdot)}(\Omega)$ and $r_1 \leq s \leq p_1$.

Now, we let

$$H(t) := -E(t)$$

and denote by C a generic positive constant that depends on Ω only. Combination of (4.4) and (4.6) leads to

Corollary 4.2 *Under the assumptions of Lemma 4.1, we have*

$$\varrho_{p(\cdot)}^{\frac{s}{p_1}}(u) \leq C\big(|H(t)| + \|u_t\|_2^2 + \varrho_{p(\cdot)}(u)\big), \tag{4.8}$$

for any $u \in W_0^{1,r(\cdot)}(\Omega)$ *and* $r_1 \leq s \leq p_1$.

As a special case, we get the following

Corollary 4.3 *Under the assumptions of Lemma 4.1, we have*

$$\|u\|_{p_1}^s \leq C\big(|H(t)| + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big), \tag{4.9}$$

for any $u \in W_0^{1,r(\cdot)}(\Omega)$ *and* $r_1 \leq s \leq p_1$.

Lemma 4.2 *Assume that (4.2),(4.3) hold and* $E(0) < 0$. *Then the solution of problem* (4.1) *satisfies, for some* $c > 0$,

$$\varrho_{p(\cdot)}(u) \geq c\|u\|_{p_1}^{p_1}. \tag{4.10}$$

Proof.

$$\varrho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx = \int_{\Omega_+} |u|^{p(x)} dx + \int_{\Omega_-} |u|^{p(x)} dx,$$

where

$$\Omega_+ = \{x \in \Omega \,/\, |u(x,t)| \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega \,/\, |u(x,t)| < 1\}.$$

So, we get

$$\varrho_{p(\cdot)}(u) \geq \int_{\Omega+} |u|^{p_1} + \int_{\Omega_-} |u|^{p_2} \geq \int_{\Omega+} |u|^{p_1} + c_1\Big(\int_{\Omega-} |u|^{p_1}\Big)^{\frac{p_2}{p_1}}.$$

This gives

$$c_2\big(\varrho_{p(\cdot)}(u)\big)^{\frac{p_1}{p_2}} \geq \int_{\Omega-} |u|^{p_1} \;\; \text{and} \;\; \varrho_{p(\cdot)}(u) \geq \int_{\Omega+} |u|^{p_1}$$

and, hence,

$$c_2\big(\varrho_{p(\cdot)}(u)\big)^{\frac{p_1}{p_2}} + \varrho_{p(\cdot)}(u) \geq \|u\|_{p_1}^{p_1}. \tag{4.11}$$

Since

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\varrho_{p(\cdot)}(u),$$

then (4.11) leads to

$$\varrho_{p(\cdot)}(u)\Big[1 + c_2\Big(\frac{p_1}{b}H(0)\Big)^{\frac{p_1}{p_2}-1}\Big] \geq \|u\|_{p_1}^{p_1}.$$

Thus, (4.10) follows. ∎

Lemma 4.3 *Suppose that (4.3) holds and let u be the solution of (4.1). Then,*

$$\int_\Omega |u|^{m(x)} dx \leq C\Big(\big(\varrho_{p(\cdot)}(u)\big)^{\frac{m_1}{p_1}} + \big(\varrho_{p(\cdot)}(u)\big)^{\frac{m_2}{p_1}}\Big). \tag{4.12}$$

Proof.

$$\begin{aligned}
\int_\Omega |u|^{m(x)}dx &\leq \int_{\Omega_-} |u|^{m_1}dx + \int_{\Omega_+} |u|^{m_2}dx \\
&\leq C\Big[\Big(\int_{\Omega_-} |u|^{p_1}dx\Big)^{m_1/p_1} + \Big(\int_{\Omega_+} |u|^{p_1}dx\Big)^{m_2/p_1}\Big] \\
&\leq C\big(\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2}\big) \\
&\leq C\Big(\big(\varrho_{p(\cdot)}(u)\big)^{\frac{m_1}{p_1}} + \big(\varrho_{p(\cdot)}(u)\big)^{\frac{m_2}{p_1}}\Big),
\end{aligned}$$

by Lemma 4.2. ∎

Lemma 4.4 *let u be the solution of (4.1), with $E(0) < 0$. Then there exists a constant $c_1 > 0$ such that*

$$\|\nabla u(\cdot,t)\|_{r(\cdot)} \geq c_1, \quad \forall t \geq 0. \tag{4.13}$$

Proof. Suppose, by contradiction, there exists a sequence t_k such that

$$\|\nabla u(\cdot,t_k)\|_{r(\cdot)} \to 0 \quad \text{as} \quad k \to \infty.$$

Then, Lemma 2.8 and Lemma 2.4 imply

$$\varrho_{p(\cdot)}(u(\cdot,t_k)) \to 0 \quad \text{as} \quad k \to \infty.$$

This yields

$$\lim_{k\to\infty} E(t_k) \geq 0,$$

which contradicts the fact that $E(t) \leq E(0) < 0$, $\forall t \geq 0$. ∎

4.3 Blow-Up Result For Negative Initial Energy

In this section we present our first blow up result and its proof.

Theorem 4.4 *Let the assumptions of Proposition 4.1 be satisfied and assume that*

$$E(0) < 0. \tag{4.14}$$

Then the solution of (4.1) blows up in finite time.

Proof. We multiply equation (4.1) by u_t and integrate over the domain Ω to get

$$E'(t) = -a \int_{\Omega} |u_t(x,t)|^{m(x)} dx \leq 0, \tag{4.15}$$

for a.e. $t \in [0,T)$ since E is an absolutely continuous function (see [34]); consequently, $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1} \varrho_{p(\cdot)}(u), \tag{4.16}$$

for all t in $[0,T)$, by recalling (4.14). We next define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t(x,t) dx, \tag{4.17}$$

for a small ε to be specified later and for

$$0 < \alpha \leq \min\left\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}\right\}. \tag{4.18}$$

We differentiate (4.17) and use the equation in (4.1) to arrive at

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega u_t^2 - \varepsilon\int_\Omega |\nabla u|^{r(x)} + \varepsilon b\int_\Omega |u|^{p(x)} \\ &- a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}. \end{aligned} \tag{4.19}$$

We add and subtract $\varepsilon(1-\eta)p_1 H(t)$, for $0 < \eta < 1$, in the right side of (4.19), to obtain

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1 H(t) + \varepsilon b\eta\int_\Omega |u|^{p(x)} \\ &+ \varepsilon\Big(\frac{(1-\eta)p_1}{2} + 1\Big)\|u_t\|_2^2 + \varepsilon\Big(\frac{(1-\eta)p_1}{r_2} - 1\Big)\int_\Omega |\nabla u|^{r(x)} - a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}dx. \end{aligned} \tag{4.20}$$

Then, for η small enough, we have

$$\begin{aligned} L'(t) &\geq \varepsilon\beta\big[H(t) + \|u_t\|_2^2 + \varrho_{r(\cdot)}(\nabla u) + \varrho_{p(\cdot)}(u)\big] + (1-\alpha)H^{-\alpha}(t)H'(t) \\ &- a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}dx, \end{aligned} \tag{4.21}$$

where

$$\beta = \min\left\{(1-\eta)p_1, b\eta, \frac{(1-\eta)p_1}{2} + 1, \frac{(1-\eta)p_1}{r_2} - 1\right\} > 0.$$

Now, by using Young's inequality, we estimate the last term in (4.21) as follows

$$\int_\Omega |u_t|^{m(x)-1}|u|dx \le \frac{1}{m_1}\int_\Omega \delta^{m(x)}|u|^{m(x)}dx + \frac{m_2-1}{m_2}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx, \quad \forall \delta > 0. \tag{4.22}$$

So, by choosing δ such that

$$\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),$$

where k is a large constant to be specified later, and substituting in (4.22), we obtain

$$\int_\Omega |u_t|^{m(x)-1}|u|dx \le \frac{1}{m_1}\int_\Omega k^{1-m(x)}|u|^{m(x)}H^{\alpha(m(x)-1)}(t)dx + \frac{(m_2-1)k}{am_2}H^{-\alpha}(t)H'(t). \tag{4.23}$$

Combining (4.21) and (4.23) yields

$$\begin{aligned} L'(t) \ge\ & \varepsilon\beta\big[H(t) + \|u_t\|_2^2 + \varrho_{r(\cdot)}(\nabla u) + \varrho_{p(\cdot)}(u)\big] + \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2-1}{m_2}\Big)k\Big]H^{-\alpha}(t)H'(t) \\ & - \varepsilon\frac{k^{1-m_1}a}{m_1}C_1H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}dx. \end{aligned} \tag{4.24}$$

By using (4.16) and Lemma 4.3, we get

$$H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}dx \le C\Big[\big(\varrho(u)\big)^{\frac{m_1}{p_1}+\alpha(m_2-1)} + \big(\varrho(u)\big)^{\frac{m_2}{p_1}+\alpha(m_2-1)}\Big]. \tag{4.25}$$

We then use (4.18) and Lemma 4.1, for

$$s = m_2 + \alpha p_1(m_2 - 1) \leq p_1 \text{ and } s = m_1 + \alpha p_1(m_2 - 1) \leq p_1,$$

to deduce, from (4.25), that

$$H^{\alpha(m_2-1)}(t) \int_\Omega |u|^{m(x)} dx \leq C\Big(\|\nabla u\|_{r(\cdot)}^{r_1} + \varrho_{p(\cdot)}(u) \Big). \tag{4.26}$$

By exploiting Lemma 4.4, we get

$$\|\nabla (u/c_1)\|_{r(\cdot)} \geq 1. \tag{4.27}$$

Lemma 2.4 and (4.27) leads to

$$\varrho_{r(\cdot)}\big(\nabla(u/c_1)\big) \geq \min\left\{ \|\nabla(u/c_1)\|_{r(\cdot)}^{r_1}, \|\nabla(u/c_1)\|_{r(\cdot)}^{r_2} \right\} = \|\nabla(u/c_1)\|_{r(\cdot)}^{r_1}. \tag{4.28}$$

Therefore (4.28) takes the form

$$\varrho_{r(\cdot)}\big(\nabla u\big) \geq c_2 \|\nabla u\|_{r(\cdot)}^{r_1}. \tag{4.29}$$

Combination of (4.24), (4.26) and (4.29) leads to

$$\begin{aligned} L'(t) \geq & \Big[(1-\alpha) - \varepsilon\Big(\frac{m_2 - 1}{m_2}\Big) k \Big] H^{-\alpha}(t) H'(t) \\ & + \varepsilon\Big(\beta - \frac{k^{1-m_1} a}{m_1} C\Big) \big[H(t) + \|u_t\|_2^2 + \|\nabla u\|_{r(\cdot)}^{r_1} + \varrho_{p(\cdot)}(u) \big]. \end{aligned} \tag{4.30}$$

Now, we pick k so large that

$$\gamma = \beta - \frac{ak^{1-m_1}}{m_1} C > 0.$$

Once k is chosen (hence γ), we select ε so small that

$$(1-\alpha) - \varepsilon\Big(\frac{m_2 - 1}{m_2}\Big)k \geq 0 \quad \text{and} \quad L(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0(x)u_1(x)dx > 0.$$

Therefore (4.30) takes the form

$$L'(t) \geq \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + + \|\nabla u\|_{r(\cdot)}^{r_1} + \varrho_{p(\cdot)}(u)\big] \geq \gamma\varepsilon\big[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\big], \quad (4.31)$$

by virtue of (4.10). Consequently, we have

$$L(t) \geq L(0) > 0, \quad \text{for all } \ t \geq 0.$$

Next, we want to obtain an inequality of the form

$$L'(t) \geq \Gamma L^{\frac{1}{1-\alpha}}(t), \quad \text{for all } \ t \geq 0, \quad (4.32)$$

for a positive Γ which depends only on $\varepsilon\gamma$ and C (here C is the constant of Corollary 4.1). Once (4.32) is proved, one can obtain, in a standard way, the finite time blowup of the functional $L(t)$. To prove (4.32), we first note that

$$\Big|\int_\Omega uu_t dx\Big| \le \|u\|_2\|u_t\|_2 \le C\|u\|_{p_1}\|u_t\|_2.$$

This gives

$$\Big|\int_\Omega uu_t dx\Big|^{\frac{1}{1-\alpha}} \le C\|u\|_{p_1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Again Young's inequality yields

$$\Big|\int_\Omega uu_t dx\Big|^{1/(1-\alpha)} \le C\Big[\|u\|_{p_1}^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\Big], \tag{4.33}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. Taking $\theta = 2(1-\alpha)$, we get $\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \le p_1$ by virtue of (4.18). Therefore (4.33) takes the form

$$\Big|\int_\Omega uu_t dx\Big|^{1/(1-\alpha)} \le C\Big[\|u\|_{p_1}^{s} + \|u_t\|_2^{2}\Big],$$

where $s = \frac{2}{1-2\alpha} \le p_1$. By recalling Corollary 4.3, we get

$$\Big|\int_\Omega uu_t dx\Big|^{1/(1-\alpha)} \le C\Big[H(t) + \|u\|_{p_1}^{p_1} + \|u_t\|_2^{2}\Big], \quad \forall t \ge 0. \tag{4.34}$$

Finally, we use the inequality

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \Big[H^{(1-\alpha)}(t) + \varepsilon\int_\Omega uu_t dx\Big]^{1/(1-\alpha)} \\ &\le 2^{1/(1-\alpha)}\Big[H(t) + \Big|\int_\Omega uu_t\Big|^{1/(1-\alpha)}\Big] \end{aligned}$$

and combine with (4.31) and (4.34), the inequality (4.32) is established. A simple integration of (4.32) over $(0,t)$ then yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}. \tag{4.35}$$

Therefore (4.35) shows that $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}, \tag{4.36}$$

where Γ and α are positive constant with $\alpha < 1$ and L is given by (4.17) above. This completes the proof. ∎

Remark 4.2 The estimate (4.36) shows that the larger $L(0)$ is, the quicker the blow up takes place.

4.4 Blow-Up Result For Positive Initial Energy

In this section, we establish the blow up for certain solutions with positive energy. In order to state and prove our result, let B be the best constant of the Sobolev embedding $W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and set

$$B_1 = \max\left\{1, B, \left(\frac{1}{b}\right)^{1/r_2}\right\}, \quad \alpha_1 = \left(\frac{1}{bB_1^{p_1}}\right)^{r_2/(p_1-r_2)}, \quad \alpha_0 = \|\nabla u_0\|_{r(\cdot)}^{r_2}, \quad E_1 = \left(\frac{1}{r_2} - \frac{1}{p_1}\right)\alpha_1,$$

$$H(t) = E_1 - E(t), \tag{4.37}$$

$$F(t) = H^{1-\lambda}(t) + \varepsilon \int_\Omega u u_t dx, \tag{4.38}$$

where $\varepsilon > 0$, $0 < \lambda < 1$ are to be determined later.

Our main result of this section is the following theorem.

Theorem 4.5 *Let the conditions of Proposition 4.1 be fulfilled. Assume further that*

$$E(0) < E_1, \quad \alpha_1 < \alpha_0 \leq B_1^{-r_2}. \tag{4.39}$$

Then the solution of (4.1) blows up in a finite time.

To prove the theorem, we need the following two lemmas.

Lemma 4.5 *Let the assumptions in Theorem 4.5 be satisfied, then there exists a constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u(\cdot,t)\|_{r(\cdot)}^{r_2} \geq \alpha_2 \ \ \forall t \geq 0. \tag{4.40}$$

Proof. Recalling (4.4), we have

$$\begin{aligned}
E(t) &\geq \frac{1}{r_2}\varrho_{r(\cdot)}(\nabla u) - \frac{b}{p_1}\varrho_{p(\cdot)}(u) \\
&\geq \frac{1}{r_2}\min\left\{\|\nabla u\|_{r(\cdot)}^{r_1}, \|\nabla u\|_{r(\cdot)}^{r_2}\right\} - \frac{b}{p_1}\max\left\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right\} \\
&\geq \frac{1}{r_2}\min\left\{\|\nabla u\|_{r(\cdot)}^{r_1}, \|\nabla u\|_{r(\cdot)}^{r_2}\right\} - \frac{b}{p_1}\max\left\{\left(B_1\|\nabla u\|_{r(\cdot)}\right)^{p_1}, \left(B_1\|\nabla u\|_{r(\cdot)}\right)^{p_2}\right\} \\
&\geq \frac{1}{r_2}\min\left\{\alpha^{\frac{r_1}{r_2}}, \alpha\right\} - \frac{b}{p_1}\max\left\{\left(B_1^{r_2}\alpha\right)^{\frac{p_1}{r_2}}, \left(B_1^{r_2}\alpha\right)^{\frac{p_2}{r_2}}\right\} := g(\alpha), \ \ \forall \alpha \in [0,\infty),
\end{aligned}$$

where $\alpha = \|\nabla u\|_{r(\cdot)}^{r_2}$. Let

$$h(\alpha) = \frac{1}{r_2}\alpha - \frac{b}{p_1}\left(B_1^{r_2}\alpha\right)^{\frac{p_1}{r_2}}.$$

Notice that $h(\alpha) = g(\alpha)$, for $0 < \alpha \leq B_1^{-r_2}$. It is easy to check that the function $h(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha < +\infty$.

Since $E(0) < E_1 = h(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, \infty)$ such that $h(\alpha_2) = E(0)$. Then, we have $h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2)$. It implies that $\alpha_0 \geq \alpha_2$.

Now, to prove (4.40), we suppose on the contrary that $\|\nabla u(t_0)\|_{r(\cdot)}^{r_2} < \alpha_2$, for some $t_0 > 0$. So, there exists $t_1 > 0$ such that $\alpha_1 < \|\nabla u(t_1)\|_{r(\cdot)}^{r_2} < \alpha_2$. Using the monotonicity of $h(\alpha)$, we have

$$E(t_1) \geq h\left(\|\nabla u(t_1)\|_{r(\cdot)}^{r_2}\right) > h(\alpha_2) = E(0),$$

which contradicts $E(t) < E(0)$, for all $t \in (0, T)$. Thus, (4.40) is established. ∎

Lemma 4.6 *Let the assumptions in Theorem 4.5 be satisfied, then we have*

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\varrho_{p(\cdot)}(u).$$

Proof. Using (4.4),(4.15) and (4.37), we obtain

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &\leq E_1 - \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} dx\right] + b\int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx \end{aligned}$$

and, from (4.40), we get

$$\begin{aligned} E_1 - \left[\frac{1}{2}\int_\Omega u_t^2 dx + \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)} dx\right] &\leq E_1 - \frac{1}{r_2}\int_\Omega |\nabla u|^{r(x)} dx \\ &\leq E_1 - \frac{1}{r_2}\min\left\{\|\nabla u\|_{r(\cdot)}^{r_1}, \|\nabla u\|_{r(\cdot)}^{r_2}\right\} \\ &\leq E_1 - \frac{1}{r_2}\min\left\{\alpha_2^{\frac{r_1}{r_2}}, \alpha_2\right\} \\ &\leq E_1 - \frac{1}{r_2}\min\left\{\alpha_1^{\frac{r_1}{r_2}}, \alpha_1\right\} \\ &= E_1 - \frac{1}{r_2}\alpha_1 = -\frac{\alpha_1}{p_1} < 0, \quad \forall t \geq 0. \end{aligned}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{b}{p_1}\varrho_{p(\cdot)}(u), \quad \forall t \geq 0.$$

∎

Proof of Theorem 4.5. With the help of Lemma 4.6, the proof is established exactly by repeating the steps (4.17)-(4.34) of the proof of Theorem 4.4.

CHAPTER 5

A BLOW-UP RESULT OF ARBITRARY POSITIVE-INITIAL ENERGY SOLUTIONS OF A NONLINEAR WAVE EQUATION WITH VARIABLE EXPONENTS

In this chapter, we investigate the long-time behavior of a nonlinear wave equation with variable exponents. We use the modified concavity method to establish the blow up result of the problem. To this end, we consider the following initial-boundary-value problem:

$$\begin{cases} u_{tt} - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) + \mu u_t = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\mu \geq 0$ is a constant, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and the exponents m and p are given measurable functions defined on $\overline{\Omega}$ satisfying some conditions to be specified later. The rest of the chapter is organized as follows. In Section 5.1, we introduce some assumptions needed in this chapter. An important lemma and the statement without proof of the well-posedness of our problem will be given in Section 5.2. In Section 5.3, we give the main blow-up result.

5.1 Assumptions

In this section, we present some materials needed in the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the variable-exponent Sobolev space $W_0^{1,m(\cdot)}(\Omega)$ with their norms. We assume the following hypotheses

$(A1)$ The exponents m and p are measurable functions such that either $m, p \in$

$C(\overline{\Omega})$ or they satisfy the following log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{5.2}$$

$A > 0, \quad 0 < \delta < 1.$

$(A2)$ m and p satisfy the following condition

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < m_*(x), \tag{5.3}$$

where

$$m_*(x) = \begin{cases} \frac{nm(x)}{\operatorname{esssup}_{x\in\Omega}(n-m(x))} & \text{if } m_2 < n \\ +\infty & \text{if } m_2 \geq n \end{cases}$$

and

$$\operatorname{essinf}_{x\in\Omega}(m^*(x) - p(x)) > 0.$$

5.2 The Well-posedness of the Problem

In this section we give the statement without proof of the wellposedness of our problem (5.1) as well as a lemma which needed for the proof of our main result. First, we introduce the following energy functional

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 dx + \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)} dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx, \ \ t \geq 0. \tag{5.4}$$

Proposition 5.1 [11] *Let* $u_0 \in W_0^{1,m(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$ *and* $m, p \in \mathcal{C}(\overline{\Omega})$ *satisfy condition* (*A*2), *the problem (5.1) has a unique weak solution such that*

$$u \in L^\infty\Big((0,T), W_0^{1,m(\cdot)}(\Omega)\Big), \quad u_t \in L^\infty\Big((0,T), L^2(\Omega)\Big), \quad u_{tt} \in L^\infty\Big((0,T), W^{-1,m'(\cdot)}(\Omega)\Big),$$

where $\frac{1}{m(\cdot)} + \frac{1}{m'(\cdot)} = 1$.

Remark 5.1 *The proof of this proposition can be established employing the Galerkin method and a fixed-point argument as in chapter 3. See also [12].*

Lemma 5.1 [58] *Assume that* $\Phi \in C^2([0,T))$ *satisfying*

$$\Phi\Phi_{tt} - \alpha\Phi_t^2 + \gamma\Phi\Phi_t + \beta\Phi \geq 0, \quad \alpha > 1, \beta \geq 0, \gamma \geq 0, \tag{5.5}$$

$$\Phi(t) \geq 0, \quad \Phi(0) > 0,$$

and

$$\Phi_t(0) > \frac{\gamma}{\alpha-1}\Phi(0), \quad \Big(\Phi_t(0) - \frac{\gamma}{\alpha-1}\Phi(0)\Big)^2 > \frac{2\beta}{2\alpha-1}\Phi(0). \tag{5.6}$$

Then,

$$\limsup_{t\to T} \Phi(t) = +\infty,$$

where

$$T \leq \frac{\Phi^{1-\alpha}(0)}{A}, \quad A^2 = (\alpha-1)^2\Phi^{-2\alpha}(0)\Big[\Big(\Phi_t(0) - \frac{\gamma}{\alpha-1}\Phi(0)\Big)^2 - \frac{2\beta}{2\alpha-1}\Phi(0)\Big]. \tag{5.7}$$

Moreover, $\Phi(t)$ *satisfies*

$$\Phi(t) \geq \frac{e^{\frac{\gamma t}{\alpha-1}}}{[\Phi^{1-\alpha}(0) - At]^{\frac{1}{\alpha-1}}}.$$

5.3 The Main Result

In this section we state and prove our main result.

Theorem 5.1 *Let the conditions of Proposition 5.1 be fulfilled. Assume further that the initial data are such that* $E(0) > 0$,

$$\int_\Omega u_0 u_1 dx > \frac{2\mu}{p_1 - 2}\|u_0\|_2^2 > 0,$$

and

$$\Big(\int_\Omega u_0 u_1 dx - \frac{2\mu}{p_1 - 2}\|u_0\|_2^2\Big)^2 > 2E(0)\|u_0\|_2^2,$$

then there exists

$$T^* \leq \frac{1}{A}\|u_0\|_2^{1-\frac{p_1}{2}} \quad \textit{such that} \quad \limsup_{t\to T^*}\|u(\cdot,t)\|_2 = +\infty,$$

where

$$A^2 = \Big(1 - \frac{p_1}{2}\Big)^2 \|u_0\|_2^{-2-p_1}\Big[\Big(\int_\Omega u_0 u_1 dx - \frac{2\mu}{p_1 - 2}\|u_0\|_2^2\Big)^2 - 2E(0)\|u_0\|_2^2\Big]. \quad (5.8)$$

Proof. We multiply (5.1) by u and integrate over Ω to obtain

$$\int_\Omega uu_{tt} + \int_\Omega |\nabla u|^{m(x)}dx + \frac{\mu}{2}\frac{d}{dt}\int_\Omega |u|^2 = \int_\Omega |u|^{p(x)}dx. \tag{5.9}$$

Let

$$\Phi(t) = \int_\Omega |u|^2 dx, \qquad J(t) = \int_\Omega |u_t|^2 dx.$$

Then,

$$\int_\Omega uu_{tt} = \frac{1}{2}\frac{d^2\Phi}{dt^2} - J(t). \tag{5.10}$$

Combination of (5.9) and (5.10) yield

$$\frac{1}{2}\frac{d^2\Phi}{dt^2} - J(t) + \frac{\mu}{2}\frac{d\Phi}{dt} + \int_\Omega |\nabla u|^{m(x)}dx = \int_\Omega |u|^{p(x)}dx. \tag{5.11}$$

Similarly, we multiply (5.1) by u_t and integrate over Ω to get

$$\frac{d}{dt}\Big[\frac{1}{2}J + \int_\Omega \frac{1}{m(x)}|\nabla u|^{m(x)}dx\Big] + \mu J = \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx. \tag{5.12}$$

Integration of (5.12) over $(0,t)$, dropping

$$\int_0^t \mu J(t)dt \geq 0,$$

gives

$$\frac{1}{2}J + \frac{1}{m_2}\int_\Omega |\nabla u|^{m(x)}dx - E(0) \leq \frac{1}{p_1}\int_\Omega |u|^{p(x)}dx, \tag{5.13}$$

where

$$E(0) = \frac{1}{2}\int_\Omega |u_1|^2 dx + \int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx - \int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx.$$

Combining (5.11) and (5.13), we arrive at

$$\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} + p_1 E(0) \geq \Big(1 + \frac{p_1}{2}\Big)J + \Big(\frac{p_1}{m_2} - 1\Big)\int_\Omega |\nabla u|^{m(x)}dx. \tag{5.14}$$

As $m_2 \leq p_1$, we may drop the last term of (5.14) to get

$$\frac{1}{2}\frac{d^2\Phi}{dt^2} + \frac{\mu}{2}\frac{d\Phi}{dt} + p_1 E(0) \geq \Big(1 + \frac{p_1}{2}\Big)J.$$

Then, Cauchy-Schwarz inequality yields

$$\Phi_t^2 \leq 4J\Phi.$$

Combining the last two inequalities, we obtain

$$\Phi\Phi_{tt} - \frac{1}{2}\Big(1 + \frac{p_1}{2}\Big)\Phi_t^2 + \mu\Phi\Phi_t + 2p_1 E(0)\Phi \geq 0. \tag{5.15}$$

Comparing (5.15) with (5.5), we easily see that

$$\alpha = \frac{1}{2}\Big(1 + \frac{p_1}{2}\Big) > 1, \quad \beta = 2p_1 E(0) > 0, \quad \gamma = \mu \geq 0,$$

$$\frac{2\beta}{2\alpha-1} = 8E(0), \qquad \frac{\gamma}{\alpha-1} = \frac{4\mu}{p_1-2}. \tag{5.16}$$

Thus, by Lemma 5.1, there exists

$$T^* \leq \frac{1}{A}\|u_0\|_2^{1-\frac{p_1}{2}} \quad \text{such that} \quad \limsup_{t\to T^*}\|u(\cdot,t)\|_2 = +\infty, \tag{5.17}$$

where A is given by (5.8). Moreover, we have

$$\|u_0\|_2 \geq \frac{e^{\frac{2\mu t}{p_1-2}}}{\left[\|u_0\|_2^{1-\frac{p_1}{2}} - At\right]^{\frac{2}{p_1-2}}},$$

which is the desired result.

Remark 5.2 This result extends the one in [58] to problems with variable-exponent nonlinearities.

Remark 5.3 Conditions assumed in Theorem 5.1 are compatible for $\mu \geq 0$ small enough. To show this, we fix $u_0 \in W_0^{1,m(\cdot)}(\Omega)$ large enough so that

$$\int_\Omega |u_0|^{p(x)}dx > \frac{2p_2}{p_1-2}\int_\Omega |u_0|^2 dx + \frac{p_2}{m_1}\int_\Omega |\nabla u_0|^{m(x)}dx, \tag{5.18}$$

then we choose $u_1 = \lambda u_0$, for $\lambda > \frac{2\mu}{p_1-2}$, so that

$$\Big(\Phi_t(0) - \frac{4\mu}{p_1-2}\Phi(0)\Big) = 2\Big(\lambda - \frac{2\mu}{p_1-2}\Big)\Phi(0) > 0, \tag{5.19}$$

and λ large enough (if needed), so that

$$E(0)=\frac{\lambda^2}{2}\int_\Omega |u_0|^2dx+\int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx-\int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx>0.$$

Now, using (5.19), we estimate

$$\Big(\Phi_t(0)-\frac{4\mu}{p_1-2}\Phi(0)\Big)^2=\Big(4\lambda^2-\frac{16\mu\lambda}{p_1-2}+\frac{16\mu^2}{(p_1-2)^2}\Big)\Phi^2(0)>0 \tag{5.20}$$

and

$$\begin{aligned}8E(0)\Phi(0)=\ & 8\Phi(0)\Big[\frac{\lambda^2}{2}\int_\Omega |u_0|^2dx+\int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx-\int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx\Big]\\ =\ & 4\lambda^2\Phi^2(0)+8\Phi(0)\Big[\int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx-\int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx\Big]\end{aligned}$$

Comparing the last equation and (5.20), it suffices to show that

$$\Big(-\frac{2\mu\lambda}{p_1-2}+\frac{2\mu^2}{(p_1-2)^2}\Big)\Phi(0)>\int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx-\int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx,$$

or

$$\frac{2\mu^2}{(p_1-2)^2}\int_\Omega |u_0|^2dx+\int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}dx>\frac{2\mu\lambda}{p_1-2}\int_\Omega |u_0|^2dx+\int_\Omega \frac{1}{m(x)}|\nabla u_0|^{m(x)}dx. \tag{5.21}$$

To get (5.21), we take $\lambda=\frac{1}{\mu}$ and exploit (5.18) then the last condition in Theorem 5.1 follows.

We also note here that (5.19) and $\lambda = \frac{1}{\mu}$ imply that

$$\mu \in \left(0, \sqrt{\frac{p_1}{2} - 1}\right).$$

For $\mu = 0$, it is enough to take λ large. It is clear that the smaller μ is the lager the "positive" initial energy we can take. Moreover, this gives an indication that larger damping may prevent solutions with positive energy from blowing up.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

In this work we considered three nonlinear hyperbolic problems involving nonclassical nonlinearities and obtained blow up results for solutions associated to these problems under appropriate assumptions on the exponents of nonlinearity and the initial data. We extended various blow up results from the constant-exponent case to the variable-exponent case and established some existence results for nonlinear hyperbolic problems with variable exponent nonlinearities. In particular, we extended the blow up result of some nonlinear hyperbolic problem, considered by Georgiev and Todorova [34] and Messaoudi [72], from the constant-exponent case to the variable-exponent case. Also, we extended the one in Korpusov [58], to problems with variable-exponent nonlinearities. We also verified and illustrated one blow-up result numerically.

6.2 Future Work

Investigating other nonlinear hyperbolic problems

There exist nonlinear hyperbolic problems with variable exponents that can be studied. Here are some examples:

- Extend the blow up results to some Kirchhoff-type problems involving nonstandard nonlinearities.
- Extend the blow up results to some viscoelastic problems involving nonstandard nonlinearities.
- Investigate the stability of some nonlinear wave equations involving variable exponents.
- Consider some systems with nonstandard nonlinearities.

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Publications

The following papers were Published/Submitted from our research

(1) Messaoudi S.A. and Talahmeh A.A, *A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities,* Applicable Analysis 96 no. 9 (2017), 1509-1515.

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